XXII. *On the Differential Covariants of Plane Curves, and the Operators employed in their Development.*

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The subject of the earlier parts of this paper is immediately allied with the late M. HALPHEN'S investigations in his Thesis, "Sur les Invariants Différentiels," (GAUTHIER-VILLARS, Paris, 1878), and in his Memoir, "Sur les Invariants Différentiels des courbes gauches" ('Journal de l'École Polytechnique,' vol. 28, cahier 47, 1880), and in the latter paper the word covariant is applied to a function having the character of the functions which are generally treated of in this paper.

The connection is perhaps greater with a previous, and, I think, less familiar paper of the same author, "Sur la recherche des points d'une courbe algébrique plane, qui satisfont à une condition exprimée par une équation différentielle algébrique, et sur les questions analogues dans l'espace" ('Journal de Math.,' 3rd series, vol. 2, 1876). I am not able to point out any paper which touches upon the actual investigation attempted in this paper, and the mode of investigation can not claim any resemblance to any of HALPHEN'S work, and does not offer in itself any element of novelty.

The subject of Differential Invariants has been further treated by SOPHUS LIE, "Über Differentialinvarianten" ('Mathem. Annalen,' vol. 24, p. 564, 1884), and by SYLVESTER, in his lectures "On the Theory of Reciprocants" ('Amer. Journ. Math.,' vols. 8, 9, and 10, 1886 and 1887).

The literature on the cognate subject of Reciprocants has become extensive, several papers by ELLIOTT, FORSYTH, HAMMOND, LEUDESDORF, and ROGERS have appeared in the 'Proceedings of the Mathematical Society' (vols. 17, 18, and 19), and in the 'Messenger of Mathematics' (vols. 17, 18, and 19). Of these papers, those of which the analytical method is most similar to mine are by Mr. FORSYTH, "Homographic Invariants and Quotient Derivatives" ('Mess. of Maths.,' vol. 17, No. 10, p. 154, 1888); "On a class of Functional Invariants" ('Phil. Trans.,' A, 1889, p. 71). I may also mention, as cognate matter, papers by Mr. ELLIOTT "On the Interchange of Variables in Certain Linear Differential Operators" ('Phil. Trans.,' A, 1890, p. 19), by Professor L. J. ROGERS, "Conjugate Annihilators" ('Mess. of Maths.,' vol. 18, p. 153–158, 1889), and by Major MACMAHON, "On Multilinear Partial Differential Operators" ('Lond. Math. Soc. Proc.,' vols. 18 and 19, 1887 and 1888).

Starting with the mode of formation of Differential Covariants, I attempt to give greater prominence to the geometrical character of the problem of invariancy, which seems to have been prominent in HALPHEN'S view of the subject, and hope that while the results are partly coextensive with the work of Professor SYLVESTER and others, they may still be complementary, and may be found applicable in investigating the characteristics of higher plane curves.

§ 1. *General Statement of the Problem Considered, as far as relates to Plane Curves.*

If a plane curve is transformed by the general homographic transformation, the transformed curve remains of the same order and possesses the same perspective singularities. If we introduce no other complications we may say that HALPHEN'S Differential Invariants indicate properties which, if they exist at any point of the original curve, exist at the same point of the transformed curve. Any function of the differential coefficients at any point of the curve which, a definite factor *près*, retains its form under the general homographic transformation is a differential invariant. I shall write these differential coefficients $y_1, y_2, y_3 \ldots$ In this paper I introduce a curve, the current coordinates of which I write ξ, η, and of which the equation consists of an algebraic relation between ξ, η and $x, y, y_1, y_2, y_3 \ldots$ (x, y being the coordinates of a point on the curve which I shall call the standard curve). Then, if this equation is such that, a definite factor *près*, its form is unaltered under the general homographic transformation, I call it a differential covariant of the standard curve.

Besides the sense in which a differential invariant has here been spoken of as indicating a perspective singularity at a point in a curve, it may, equated to zero, be the differential equation to a curve, each point of which has that singularity. (I here consider a sextactic point and other points of similar characteristics, as singular points for homographic projection.)

A conic is one of the curves which, by a general homographic transformation, can be transformed into itself, so that any one arbitrary point can be replaced by any other. Hence, if $\mathrm{V} = 0$ is the differential condition for a sextactic point, it is the differential invariant equation of conics, and if $f(\xi, \eta, y_1, y_2, y_3, y_4) = 0$ is a covariant equation of the second degree, such that when $\xi = x$ and $\eta = y$ we have $\eta_1 = y_1$, $\eta_2 = y_2$, $\eta_3 = y_3$, $\eta_4 = y_4$, it is the equation to the conic osculating the standard curve at (x, y). Also regarding ξ, η as the variables and $x, y, y_1 \ldots$ as arbitrary constants, it is the complete integral of $\mathrm{V} = 0$. Generally the differential covariants will include the integrals of the differential-invariantal equations, provided these integrals are rational, integral, algebraic functions.

Having found a covariant curve, its polars, as well as its ordinary covariants, will be included among the differential-covariant curves.

To find the degree of contact with the standard curve, we substitute for the covariant $\xi - x = h$, $\eta - y = y_1 h + y_2 h^2/2\,! +$ &c., if the equation is identically satisfied by $\eta_1 = y_1$, $\eta_2 = y_2 \ldots$, as far as the coefficient of h^n, the contact is of the n^{th} order. Equating to zero the coefficient of h^{n+1} we obtain the condition for contact of the $n + 1^{\text{th}}$ order. Since this condition is generally capable of being satisfied, and since the relation is unaltered by homographic transformation, the coefficient of h^{n+1} is a differential invariant.

The first section of this paper investigates the conditions which must be satisfied in order that (except as concerns a factor), the form of a function may be unaltered in consequence of any homographic transformation whatever, and finds the form of the factor.

The results are given in the equations (15).

In the second section, the relations of certain partial differential operators Ω_1, Ω_2, and Ω_3 with one another, and d/dx are considered. A method of eduction of covariants is established, and a method of development of a covariant from a source or matrix, as well as a process of eduction of matrices. The matrices are shown to be dependent upon differential invariants and two fundamental matrices u_4 and L_6, and from the mode in which these enter, the order of the covariant can be predicted.

This section prepares the way for the application of the theory in the two last sections.

The third section deals with the dual reciprocal relationship between the operators, and the deduction of reciprocal and contravariant functions.

The fourth section contains applications to the cubic of the methods which have been considered. The equation to the osculating cubic is determined when the cubic is non-singular, nodal, and cuspidal, and the differential equation in each case found.

The fifth section takes a new view of the subject. It is shown that the position of any point which is "homographically persistent" with regard to the standard curve (which, for example, is a point of intersection of two curves which are each covariant) is determined by the ratios of three differential-invariant functions. These can therefore be considered as invariantal coordinates of the point. It is also shown that there are similarly invariantal coordinates of a line, and the relations of these line and point coordinates are investigated, showing that the general methods of higher geometry are applicable to the novel system of coordinates.

The condition that the point should lie on a covariant curve is now given in terms of the invariantal coordinates by an equation in which the coefficients are the differential invariants which determine the character of the curve. For this reason I call it the intrinsic invariantal equation to the curve.

Also the invariantal coordinates are connected with the ordinary coordinates by relations having the form distinctive of a homographic transformation. And therefore, regarding the invariantal coordinates as ordinary coordinates, the curve represented by the intrinsic invariantal equation has all the perspective singularities of the original curve. No attempt is made to show the application of this method to the difficult case of the quartic, but several well known theorems regarding the cubic are shown to be arrived at without difficulty.

§ 1.

1. In general, the complete curve of the d^{th} algebraic degree is completely determined by $d(d+3)/2$ $(\equiv r)$ points, and if the coordinates of these points are known, the equation to the curve can be written down in the form of a determinant, thus

$$f(x, y, \ldots) \equiv \begin{vmatrix} \xi^d, & \xi^{d-1}\eta, & \ldots & 1 \\ x^d, & x^{d-1}y, & \ldots & 1 \\ x_1^d, & x_1^{d-1}y_1, & \ldots & 1 \\ \vdots & \vdots & & \vdots \end{vmatrix} = 0$$

If a general homographic transformation of the coordinates be now made, the form of the function will be unaltered, except by the introduction of a factor.

Denote the transformation by

$$\frac{x}{A_1X + B_1Y + C_1} = \frac{y}{A_2X + B_2Y + C_2} = \frac{1}{AX + BY + C} \quad . \quad . \quad . \quad . \quad (1)$$

with similar relations for ξ and η, except that we may retain the letters ξ, η, in the transformed equation without causing inconvenience.

Thus

$$f(x, y, \ldots) = D^{\frac{d(d+1)(d+2)}{3!}} (A\xi + B\eta + C)^{-d} (AX_1 + BY_1 + C)^{-d} \ldots \times f(X, Y, \ldots).$$

We may also write

$$f(x, y, \ldots) \equiv \begin{vmatrix} (\xi - x)^d, & (\xi - x)^{d-1}(\eta - y), & \ldots & \xi - x, & \eta - y, \\ (x_1 - x)^d, & (x_1 - x)^{d-1}(y_1 - y), & \ldots & x_1 - x, & y_1 - y, \\ \vdots & \vdots & \ldots & \vdots & \vdots \end{vmatrix}.$$

Imagine the r points now moved up into coincidence at x, y, subject to some geometrical condition, such as that of moving along an arc, free from singularities, of a curve of degree not lower than the d^{th}.

The equation to the resulting curve, which will osculate the assumed curve at the chosen point may now be deduced. Most easily, write

$$\xi_1 - x = h, \quad \eta_1 - y = y_1 h + \frac{y_2}{1.2} h^2 + \frac{y_3}{3\,!} h^3 + \ldots$$

For further simplicity, write $\pi - p$ for $\eta - y - y_1 (\xi - x)$, so that

$$(\pi - p)_1 = \frac{y_2}{1.2} h^2 + \frac{y_3}{3\,!} h^3 + \ldots$$

and transform the determinant so that, in the first row $\eta - y$ may be replaced by $\pi - p$.

To fill up the several columns in the determinant we write under $(\xi - x)^n (\pi - p)^m$ the coefficients of $h^2, h^3, \ldots$ in the expansion of $h^n (\pi - p)_1{}^m$. We thus obtain

$$f(x, y, \ldots) = \begin{vmatrix} (\xi - x)^d, & (\xi - x)^{d-1} (\pi - p), & \ldots & (\xi - x)(\pi - p), & \pi - p \\ 0, & 0, & \ldots & 0, & \frac{y_2}{2\,!} \\ 0, & 0, & \ldots & \frac{y_2}{2\,!}, & \frac{y_3}{2\,!} \end{vmatrix}$$

$$\times\, h \frac{d(d+1)(d+2)(d+3)}{8} - \frac{d(d+3)}{2}.$$

The transformed equation is affected in the same way, so that

$$f(x, y, \ldots) = \mathrm{D}^{d(d+1)(d+2)/3\,!} (\mathrm{A}\xi + \mathrm{B}\eta + \mathrm{C})^{-d} (\mathrm{AX} + \mathrm{BY} + \mathrm{C})^{-d^2(d+3)/2}$$

$$\times\, \mathrm{H}^{d(d+1)(d+2)(d+3)/8 - d(d+3)/2} \begin{vmatrix} (\xi - \mathrm{X})^d, & (\xi - \mathrm{X})^{d-1} (\pi - \mathrm{P}), & \ldots & & \pi - \mathrm{P} \\ 0, & 0, & & 0, & \frac{y_2}{2\,!} \\ 0, & 0, & & \frac{y_2}{2\,!}, & \frac{y_2}{2\,!} \end{vmatrix}.$$

Now write the determinants shortly

$$f(\xi - x, \pi - p, y_2 \ldots) \quad \text{and} \quad f(\xi - \mathrm{X}, \pi - \mathrm{P}, \mathrm{Y}_2 \ldots).$$

Also

$$\mathrm{H}/h = d\mathrm{X}/dx = \mu^2 \lambda^{-1}$$

where

$$\left.\begin{array}{l} \mathrm{AX} + \mathrm{BY} + \mathrm{C} = \mu. \\ (\mathrm{AX} + \mathrm{BY} + \mathrm{C})(\mathrm{A}_1 + \mathrm{B}_1\mathrm{Y}_1) - (\mathrm{A}_1\mathrm{X} + \mathrm{B}_1\mathrm{Y} + \mathrm{C}_1)(\mathrm{A} + \mathrm{BY}_1) = \lambda \\ \text{Also write} \\ \mathrm{A}\xi + \mathrm{B}\eta + \mathrm{C} = \nu. \end{array}\right\} \quad (2).$$

Then

$$f(\xi - x, \pi - p, y_2, \ldots) \\ = D^{d(d+1)(d+2)/3!} \lambda^{-d(d+1)(d+2)(d+3)/8+d(d+3)/2} \mu^{d(d+1)(d+2)(d+3)/4-d(d+3)-d^2(d+3)/2} \nu^{-d} \\ \times f(\xi - X, \pi - P, Y_2 \ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a).$$

To put the multiplying factor in a more concise form, we will use HALPHEN's nomenclature. If a term contains the differential coefficients $y_2, y_3 \ldots$, we will call the number of differential coefficients in the term its degree, and write it d_x, and the sum of the indices of differentiation its weight, and write it w.

Considering the transformation to be made in the ultimate form of the equation, we see that

$$\begin{aligned} \pi - p &= D\lambda^{-1}\nu^{-1}(\pi - P) \\ y_2 &= D\mu^3\lambda^{-3}Y_2 \\ y_3 &= D\mu^5\lambda^{-4}Y_3 + \&c. \\ y_k &= D\mu^{2k-1}\lambda^{-(k+1)}Y_k + \&c. \end{aligned}$$

From the mode of formation of the determinant, the degree of the terms in each coefficient of $\xi - x$ and $\pi - p$ is uniform, and their weight is uniform, so that we may consider the weight and degree of each coefficient.

Let d_x and w denote the degree and weight of the coefficient of the highest power of $\pi - p$ in the expanded determinant.

We thus determine the multiplying factor to be

$$D^{d + d_x} \lambda^{-(d + d_x + w)} \mu^{2w - d_x} \nu^{-d}.$$

From this we conclude that

$$\begin{aligned} d + d_x &= \frac{d(d+1)(d+2)}{3!} \\ w &= \frac{d(d+1)(d+2)(3d+5)}{4!} - \frac{d(d+3)}{2}, \end{aligned}$$

results which can be verified by arranging the terms in the determinant in descending order and summing the quantities required.

It is also easy to see that if $d_x(m, n)$ and $w(m, n)$ are the degree and weight of the coefficient of $(\pi - p)^{d-m}(\xi - x)^n$

$$\begin{aligned} d_x(m. n) &= d_x + m \\ w(m, n) &= w - n, \end{aligned}$$

or the determinant is a homogeneous function of $\pi - p$ and of the differential coefficients (independent of their order), and that the difference of the weight of the coefficient of a term and the index of the power of $\xi - x$ which it contains is uniform throughout.

These determine the character of the general equation of a curve of the d^{th} degree, osculating a given curve, the order of the highest differential coefficient being $d(d+3)/2 - 1$.

In its differential equation the degree is greater than this value by d, the weight by $\frac{1}{2}d(d+3)$, and the order by 1.

If the curve has any imposed condition, such as the possession of a double point, these criteria are altered.

2. *Determination of Differential Covariants of Plane Curves.*

Abandoning the condition, as yet imposed, of considering the equation to an osculating curve, the question proposed is to examine the conditions, both simple and in the form of linear partial differential equations, which $f(\xi - x, \eta - y, y_1, y_2, \ldots)$ must satisfy in order that it may become

$$D^{d+d_x} \lambda^{-(d+d_x+w)} \mu^{2w-d_x} \nu^{-d} f(\xi - X, \eta - Y, Y_1, Y_2, \ldots) \quad . \quad . \quad . \quad (36)$$

under the general linear homographic transformation stated above.

The condition that d_x and w are necessarily connected with d, by the relations just found, will not be required, but they will serve as criteria for recognising general cases.

The simple conditions are exactly analogous to those found by HALPHEN for Differential Invariants ('Thèse,' p. 21), and result from the same simple transformations.

1. From putting $x = X$, $\xi = \xi$, $y = B_2 Y$, $\eta = B_2 \eta$, we conclude, as before, that the function is homogeneous in $\eta - y$ and the several differential coefficients of y.

2. From putting $x = A_1 X$, $\xi = A_1 \xi$, $y = Y$, $\eta = \eta$, we conclude that the weight of the coefficient of each power of $\xi - x$ is uniform, and that this weight diminished by the index of the power of $\xi - x$ is uniform throughout the function.

3. We have chosen the form of the expression, so that we draw no further conclusion from a mere change of origin of coordinates. The form of the multiplying factor as well as that of the function might have been found from the method which follows, and its correctness is established from it.

To obtain the linear partial differential equations which the functions must satisfy, we consider infinitesimal transformations. Being infinitesimal we may consider them separately. The transformations are four in number. The first two would apply to a general infinitesimal Cartesian transformation, and so the results are the partial differential equations suitable to that case. The other two only concern our more general transformation, and, as we shall see, the results have a different character from those of the first two.

Write

$$x = \mathrm{X}, \qquad \xi = \xi, \qquad y = \mathrm{Y} + \epsilon\mathrm{X}, \qquad \eta = \eta + \epsilon\xi,$$

where, ϵ being infinitesimal, we shall only retain its first power.

Then, by hypothesis,

$$f(\xi - x,\ \eta - y,\ y_1, y_2, \ldots) = f(\xi - \mathrm{X},\ \eta - \mathrm{Y},\ \mathrm{Y}_1, \mathrm{Y}_2, \ldots).$$

Also expanding by TAYLOR's theorem

$$\begin{aligned} &f(\xi - x,\ \eta - y,\ y_1, y_2, \ldots) \\ &\quad = f(\xi - \mathrm{X},\ \eta - \mathrm{Y},\ \mathrm{Y}_1, \mathrm{Y}_2, \ldots) + \epsilon\left\{(\xi - \mathrm{X})\frac{\partial f}{\partial(\eta - \mathrm{Y})} + \frac{\partial f}{\partial \mathrm{Y}_1}\right\}. \end{aligned}$$

Therefore

$$(\xi - \mathrm{X})\frac{\partial f}{\partial(\eta - \mathrm{Y})} + \frac{\partial f}{\partial \mathrm{Y}_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence we come to the conclusion we arrived at before, that

$\eta - \mathrm{Y}$ and Y_1 only enter in the form $\eta - \mathrm{Y} - \mathrm{Y}_1(\xi - \mathrm{X})$, or $\pi - \mathrm{P}$.

We shall now reintroduce $\pi - p$, and write the functions

$$f(\xi - x,\ \pi - p,\ y_2, y_3, \ldots), \text{ \&c.}$$

The method we have used here we shall use in the other cases. We evaluate the multiplying factor to obtain one expression for $f(\xi - x,\ \pi - p, \ldots)$, and obtain a second by expanding the function in accordance with the transformation considered.

3. Having now introduced $\pi - p$, we shall treat the other infinitesimal transformations by a more general method.

Consider the homographic transformation

$$\frac{x}{\mathrm{X} + \mathrm{B}_1\mathrm{Y}} = \frac{y}{\mathrm{Y}} = \frac{1}{\mathrm{AX} + \mathrm{BY} + 1},$$

where B_1, A and B are indefinitely small quantities whose squares and products will be neglected.

Thus

$$\begin{aligned} \lambda &= 1 + \mathrm{B}_1\mathrm{Y}_1 + \mathrm{B}(\mathrm{Y} - \mathrm{XY}_1) \\ \mu &= 1 + \mathrm{AX} + \mathrm{BY} \\ \nu &= 1 + \mathrm{A}\xi + \mathrm{B}\eta \end{aligned}$$

so that the multiplying factor

$$\begin{aligned}&= 1 - (d + d_x + w)\{B_1Y_1 + B(Y - XY_1)\}\\ &\quad + (2w - d_x)(AX + BY)\\ &\qquad - d(A\xi + B\eta).\end{aligned}$$

4. *Determination of* Y_n.

The dependent variable is supposed to be connected with the independent by some relation, say

$$y = \phi(x),$$

so that $y_n/n!$ is the coefficient of h^n in the expansion of $\phi(x + h)$.

The homographic relations may for our present purpose be written

$$\begin{aligned}x &= X + B_1Y - X(AX + BY)\\ y &= Y - Y(AX + BY),\end{aligned}$$

whence

$$Y = \phi\{X + B_1Y - X(AX + BY)\} + Y(AX + BY)\ldots$$

where on the right-hand side Y never appears except when affected by a small coefficient.

Now write $X + h$ in place of X, and consequently

$$Y + Y_1h + \&c. + \frac{Y_m}{m!}h^m + \&c. \;(\equiv Y + \psi) \text{ for } Y,$$

the coefficient of h^n in the resulting expression on the right is then

$$Y_n/n!.$$

But this expression may be written

$$\begin{aligned}&\phi\{x + h + B_1\psi - (2AX + BY)h - BX\psi - Ah^2 - Bh\psi\}\\ &+ Y(AX + BY) + A(X\psi + Yh + h\psi) + B(2Y\psi + \psi^2).\end{aligned}$$

Hence

$$Y_1 = y_1\{1 + B_1Y_1 - BXY_1 - 2AX - BY\} + A(XY_1 + Y) + 2BYY_1,$$

and generally $Y_n/n! =$ the coefficient of h^n from

$$
\begin{aligned}
&\Sigma\, y_p h^{p-1}\,[h + p\,\{\mathrm{B}_1\psi - (2\mathrm{AX} + \mathrm{BY})\,h - \mathrm{BX}\psi - \mathrm{A}h^2 - \mathrm{B}h\psi\}\,]/p\,! \\
&\qquad + \mathrm{A}\left\{\frac{\mathrm{XY}_n}{n!} + \frac{\mathrm{Y}_{n-1}}{n-1!}\right\} \\
&\qquad + \mathrm{B}\left\{\frac{2\mathrm{YY}_n}{n!} + \Sigma\frac{\mathrm{Y}_p\mathrm{Y}_{n-p}}{p!\,n-p!}\right\} \\
&\quad = y_n\,\{1 - n\,(2\mathrm{AX} + \mathrm{BY})\}/n\,! \\
&\qquad + \mathrm{B}_1\,\Sigma\, y_p\mathrm{Y}_{n-p+1}/p - 1!\,n - p + 1! \\
&\qquad + \mathrm{A}\left\{\frac{\mathrm{XY}_n}{n!} + \frac{\mathrm{Y}_{n-1}}{n-1!} - \frac{\mathrm{Y}_{n-1}}{n-2!}\right\} \\
&\qquad + \mathrm{B}\left\{\frac{2\mathrm{YY}_n}{n!} - (p-1)\,\Sigma\frac{\mathrm{Y}_p\mathrm{Y}_{n-p}}{p!\,n-p!} - \mathrm{X}\,\Sigma\frac{y_p\mathrm{Y}_{n-p+1}}{p-1!\,n-p+1!}\right\}.
\end{aligned}
$$

In the small terms we may substitute Y for y, and thus obtain

$$
\begin{aligned}
y_n = \mathrm{Y}_n &- \mathrm{B}_1\,\Sigma\,{}_n\mathrm{C}_{p-1}\mathrm{Y}_p\mathrm{Y}_{n-p+1} \\
&+ \mathrm{A}\,\{(2n-1)\,\mathrm{XY}_n + n\,(n-2)\,\mathrm{Y}_{n-1}\} \\
&+ \mathrm{B}\,\{(n-2)\,\mathrm{YY}_n + \Sigma\,(p-1)\,{}_n\mathrm{C}_p\mathrm{Y}_p\mathrm{Y}_{n-p} + \mathrm{X}\,\Sigma\,{}_n\mathrm{C}_{p-1}\mathrm{Y}_p\mathrm{Y}_{n-p+1}\} \quad . \quad . \quad (5),
\end{aligned}
$$

where by Σ we denote that all possible values of p are to be taken from 1 upwards.

5. *Determination of* $\pi - p$ *and* $\xi - x$.

$\pi - p$ stands in place of $\eta - y - y_1\,(\xi - x)$, and, upon transformation, $\eta - y$ becomes

$$
\begin{aligned}
\eta - \mathrm{Y} - \mathrm{A}\,&\{(\xi - \mathrm{X})\,(\eta - \mathrm{Y}) + \mathrm{X}\,(\eta - \mathrm{Y}) + \mathrm{Y}\,(\xi - \mathrm{X})\} \\
&- \mathrm{B}\,\{(\eta - \mathrm{Y})^2 + 2\mathrm{Y}\,(\eta - \mathrm{Y})\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),
\end{aligned}
$$

$\xi - x$ becomes

$$
\begin{aligned}
\xi - \mathrm{X} &+ \mathrm{B}_1\,(\eta - \mathrm{Y}) \\
&- \mathrm{A}\,\{(\xi - \mathrm{X})^2 + 2\mathrm{X}\,(\xi - \mathrm{X})\} \\
&- \mathrm{B}\,\{(\xi - \mathrm{X})\,(\eta - \mathrm{Y}) + \mathrm{X}\,(\eta - \mathrm{Y}) + \mathrm{Y}\,(\xi - \mathrm{X})\} \quad . \quad . \quad . \quad (7),
\end{aligned}
$$

and y_1 becomes

$$
\mathrm{Y}_1 - \mathrm{B}_1\mathrm{Y}_1{}^2 + \mathrm{A}\,(\mathrm{XY}_1 - \mathrm{Y}) + \mathrm{BY}_1\,(\mathrm{XY}_1 - \mathrm{Y}) \quad . \quad . \quad . \quad . \quad . \quad (8),
$$

therefore $\pi - p$ becomes

$$
\begin{aligned}
\pi - \mathrm{P} - \mathrm{B}_1\mathrm{Y}_1\,(\pi - \mathrm{P}) - \mathrm{A}\,&\{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{X}\,(\pi - \mathrm{P})\} \\
&- \mathrm{B}\,\{(\pi - \mathrm{P})^2 + \mathrm{Y}_1\,(\pi - \mathrm{P})\,(\xi - \mathrm{X}) + (2\mathrm{Y} - \mathrm{XY}_1)\,(\pi - \mathrm{P})\} \quad . \quad (9),
\end{aligned}
$$

and $\xi - x$ becomes

$$
\begin{aligned}
\xi - \mathrm{X} &+ \mathrm{B}_1\,\{(\pi - \mathrm{P}) + \mathrm{Y}'\,(\xi - \mathrm{X})\} \\
&- \mathrm{A}\,\{(\xi - \mathrm{X})^2 + 2\mathrm{X}\,(\xi - \mathrm{X})\} \\
&- \mathrm{B}\,\{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{Y}_1\,(\xi - \mathrm{X})^2 + \mathrm{X}\,(\pi - \mathrm{P}) + (\mathrm{Y} + \mathrm{XY}_1)\,(\xi - \mathrm{X})\} \quad . \quad (10).
\end{aligned}
$$

6. *Deduction of the conditions in the form of linear partial differential equations.*

For this purpose, it will be convenient to consider the independent parts of the conditions separately, and the headings of the paragraphs indicate the portions considered.

B_1 only.

Multiplying factor $= 1 - (d + d_x + w) B_1 Y_1$

$$y_n = Y_n - B_1 \Sigma_n C_{p-1} Y_p Y_{n-p+1},$$

$$\pi - p \text{ becomes } \pi - P - B_1 Y_1 (\pi - P);$$

$$\xi - x \text{ becomes } \xi - X + B_1 \{\pi - P + Y_1 (\xi - X)\}.$$

Therefore

$$-Y_1 (\pi - P) \frac{\partial f}{\partial (\pi - P)} + \{\pi - P + Y_1 (\xi - X)\} \frac{\partial f}{\partial (\xi - X)}$$
$$= \Sigma\Sigma_n C_{p-1} Y_p Y_{n-p+1} \frac{\partial f}{\partial Y_n} - (d + d_2 + w) Y_1 f.$$

Since Y_1 does not occur explicitly in the function, we must equate separately the parts which contain Y_1 and those which do not contain it. The symbol S will now be used instead of Σ to denote that Y_1 does not occur within the summation.

Thus

$$(\pi - P) \frac{\partial f}{\partial (\pi - P)} - (\xi - X) \frac{\partial f}{\partial (\xi - X)} = (d + d_x + w) f - S (n + 1) Y_n \frac{\partial f}{\partial Y_n}$$

$$(\pi - P) \frac{\partial f}{\partial (\xi - X)} = SS_n C_{p-1} Y_p Y_{n-p+1} \frac{\partial f}{\partial Y_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Since f is homogeneous of degree $d + d_x$ in $\pi - P$ and the differential coefficients, we may write the first of these equations

$$\omega f + (\xi - X) \frac{\partial f}{\partial (\xi - X)} = S n Y_n \frac{\partial f}{\partial Y_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

which expresses analytically a condition already found.

There is no other condition necessary in order that a function may be a covariant for the ordinary Cartesian transformation, which leaves the relations of a curve with infinity unaltered, that is for a change of coordinates.

To proceed, we have

A only.

Multiplying factor

$$= 1 + (2w - d - d_x)\,\mathrm{AX} - d\mathrm{A}\,(\xi - \mathrm{X}),$$

$$y_n = \mathrm{Y}_n + \mathrm{A}\,\{(2n-1)\,\mathrm{XY}_n + n\,(n-2)\,\mathrm{Y}_{n-1}\},$$
$$\xi - x = \xi - \mathrm{X} - \mathrm{A}\,\{(\xi - \mathrm{X})^2 + 2\mathrm{X}\,(\xi - \mathrm{X})\},$$
$$\pi - p = \pi - \mathrm{P} - \mathrm{A}\,\{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{X}\,(\pi - \mathrm{P})\}.$$

Hence

$$\{(\xi - \mathrm{X})^2 + 2\mathrm{X}\,(\xi - \mathrm{X})\}\frac{\partial f}{\partial\,(\xi - \mathrm{X})} + \{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{X}\,(\pi - \mathrm{P})\}\frac{\partial f}{\partial\,(\pi - \mathrm{P})}$$
$$- \Sigma\,\{(2n-1)\,\mathrm{XY}_n + n\,(n-2)\,\mathrm{Y}_{n-1}\}\frac{\partial f}{\partial \mathrm{Y}_n}$$
$$= \{d\,(\xi - \mathrm{X}) - (2w - d - d_x)\,\mathrm{X}\}\,f.$$

The only new condition arising from these terms is

$$(\xi - \mathrm{X})\left\{(\xi - \mathrm{X})\frac{\partial f}{\partial\,(\xi - \mathrm{X})} + (\pi - \mathrm{P})\frac{\partial f}{\partial\,(\pi - \mathrm{P})} - df\right\} = \mathrm{S}n\,(n-2)\,\mathrm{Y}_{n-1}\frac{\partial f}{\partial \mathrm{Y}_n} \quad . \quad (13).$$

Lastly,

B only.

Multiplying factor

$$= 1 + \mathrm{B}\,\{(w - 2d - 2d_x)\,\mathrm{Y} + (w + d + d_x)\,\mathrm{XY}_1 - d\,(\eta - \mathrm{Y})\}$$

$$y_n = \mathrm{Y}_n + \mathrm{B}\,\{(n-2)\,\mathrm{YY}_n + \Sigma(p-1)\,{}_n\mathrm{C}_p\mathrm{Y}_p\mathrm{Y}_{n-p} + \mathrm{X}\Sigma\,{}_n\mathrm{C}_{p-1}\mathrm{Y}_p\mathrm{Y}_{n-p+1}\}$$
$$\xi - x = \xi - \mathrm{X} - \mathrm{B}\,\{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{Y}_1(\xi - \mathrm{X})^2 + (\mathrm{Y} + \mathrm{XY}_1)\,(\xi - \mathrm{X}) + \mathrm{X}\,(\pi - \mathrm{P})\}$$
$$\pi - p = \pi - \mathrm{P} - \mathrm{B}\,\{(\pi - \mathrm{P})^2 + \mathrm{Y}_1\,(\pi - \mathrm{P})\,(\xi - \mathrm{X}) + (2\mathrm{Y} - \mathrm{XY}_1)\,(\pi - \mathrm{P})\}$$

therefore

$$\{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{Y}_1\,(\xi - \mathrm{X})^2 + (\mathrm{Y} + \mathrm{XY}_1)\,(\xi - \mathrm{X}) + \mathrm{X}\,(\pi \quad \mathrm{P})\}\frac{\partial f}{\partial\,(\xi - \mathrm{X})}$$
$$+ \{(\pi - \mathrm{P})^2 + \mathrm{Y}_1\,(\pi - \mathrm{P})\,(\xi - \mathrm{X}) + (2\mathrm{Y} - \mathrm{XY}_1)\,(\pi - \mathrm{P})\}\frac{\partial f}{\partial\,(\pi - \mathrm{P})}$$
$$- \Sigma\,\{(n-2)\,\mathrm{YY}_n + \Sigma\,(p-1)\,{}_n\mathrm{C}_p\mathrm{Y}_p\mathrm{Y}_{n-p} + \mathrm{X}\,\Sigma\,{}_n\mathrm{C}_{p-1}\mathrm{Y}_p\mathrm{Y}_{n-p+1}\}\frac{\partial f}{\partial \mathrm{Y}_n}$$
$$= \{d\,(\eta - \mathrm{Y}) + (2d + 2d_x - w)\,\mathrm{Y} - (w + d + d_x)\,\mathrm{XY}_1\}\,f.$$

We have again repetitions of previous conditions with the sole new condition

$$(\pi - \mathrm{P})\left\{(\pi - \mathrm{P})\frac{\partial f}{\partial\,(\pi - \mathrm{P})} + (\xi - \mathrm{X})\frac{\partial f}{\partial\,(\xi - \mathrm{X})} - df\right\} = \mathrm{SS}(p-1)\,{}_n\mathrm{C}_p\mathrm{Y}_p\mathrm{Y}_{n-p}\frac{\partial f}{\partial \mathrm{Y}_n} \quad (14).$$

The capital letters may now be conveniently replaced by the smaller type, and the equations of condition written

$$\left.\begin{array}{ll}
(\pi - p)\dfrac{\partial f}{\partial(\xi - x)} = \mathrm{SS}\,{}_n\mathrm{C}_{p-1}y_py_{n-p+1}\dfrac{\partial f}{\partial y_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . & (\mathrm{A}) \\[2ex]
(\xi - x)\left\{(\xi - x)\dfrac{\partial f}{\partial(\xi - x)} + (\pi - p)\dfrac{\partial f}{\partial(\pi - p)} - df\right\} = \mathrm{S}n\,(n-2)\,y_{n-1}\dfrac{\partial f}{\partial y_n} \;. & (\mathrm{B}) \\[2ex]
(\pi - p)\left\{(\xi - x)\dfrac{\partial f}{\partial(\xi - x)} + (\pi - p)\dfrac{\partial f}{\partial(\pi - p)} - df\right\} & \\[2ex]
\qquad\qquad = \mathrm{SS}\,(p-1)\,{}_n\mathrm{C}_py_py_{n-p}\dfrac{\partial f}{\partial y_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . & (\mathrm{C}) \\[1ex]
\text{also} & \\
wf + (\xi - x)\dfrac{\partial f}{\partial(\xi - x)} = \mathrm{S}ny_n\dfrac{\partial f}{\partial y_n} \equiv \mathrm{W}f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . & (\alpha) \\[1ex]
\text{and} & \\
(d + d_x)f = (\pi - p)\dfrac{\partial f}{\partial(\pi - p)} + \mathrm{S}y_n\dfrac{\partial f}{\partial y_n} \equiv (\pi - p)\dfrac{\partial f}{\partial(\pi - p)} + \mathrm{H}f \quad . \quad . & (\beta)
\end{array}\right\}(15),$$

of which equations the last two will be identically satisfied by any functions which follow the laws of weight and degree which the simple conditions impose upon a covariant.

We may call these the form, weight, and degree conditions, respectively.

[A precisely analogous process may be applied to the covariants of surfaces, as is in this paper used for plane curves.

If we, in this case, write $\pi - p$ to stand for

$$\zeta - z - \frac{\partial z}{\partial x}(\xi - x) - \frac{\partial z}{\partial y}(\eta - y),$$

and $c_{m,n}$ for

$$\partial^{m+n}z/\partial x^m\partial y^n\, m!\; n!\,,$$

the set of conditions corresponding to (15) are

$$(\eta - y)\frac{\partial f}{\partial(\xi - x)} = \mathrm{S}_m\mathrm{S}_n\,(m+1)\,c_{m+1,n-1}\frac{\partial f}{\partial c_{m,n}}\,,$$

$$(\xi - x)\frac{\partial f}{\partial(\eta - y)} = \mathrm{S}_m\mathrm{S}_n\,(n+1)\,c_{m-1,n+1}\frac{\partial f}{\partial c_{m,n}}\,,$$

$$(\pi - p)\frac{\partial f}{\partial(\xi - x)} = \mathrm{S}_m\mathrm{S}_n\,\mathrm{S}_p\mathrm{S}_q\,pc_{p,q}\,c_{m-p+1\ n-q}\frac{\partial f}{\partial c_{m,n}}\,,$$

$$(\pi - p)\frac{\partial f}{\partial(\eta - y)} = \mathrm{S}_m\mathrm{S}_n\,\mathrm{S}_p\mathrm{S}_q\,qc_{p,q}\,c_{m-p,n-q+1}\frac{\partial f}{\partial c_{m,n}}\,,$$

$$(\xi - x)\left\{(\xi - x)\frac{\partial f}{\partial(\xi - x)} + (\eta - y)\frac{\partial f}{\partial(\eta - y)} + (\pi - p)\frac{\partial f}{\partial(\pi - p)} - df\right\}$$

$$= \mathrm{S}_m\mathrm{S}_n\,(m + n - 2)\,c_{m-1,n}\frac{\partial f}{\partial c_{m,n}},$$

$$(\eta - y)\left\{(\xi - x)\frac{\partial f}{\partial(\xi - x)} + (\eta - y)\frac{\partial f}{\partial(\eta - y)} + (\pi - p)\frac{\partial f}{\partial(\pi - p)} - df\right\}$$

$$= \mathrm{S}_m\mathrm{S}_n\,(m + n - 2)\,c_{m,n-1}\frac{\partial f}{\partial c_{m,n}},$$

$$(\pi - p)\left\{(\xi - x)\frac{\partial f}{\partial(\xi - x)} + (\eta - y)\frac{\partial f}{\partial(\eta - y)} + (\pi - p)\frac{\partial f}{\partial(\pi - p)} - df\right\}$$

$$= \mathrm{S}_m\mathrm{S}_n\,\mathrm{S}_p\mathrm{S}_q\,(p + q - 1)\,c_{p,q}\,c_{m-p,n-q}\frac{\partial f}{\partial c_{m,n}},$$

and

$$(\pi - p)\frac{\partial f}{\partial(\pi - p)} = (d + d_z)f - \mathrm{S}_m\mathrm{S}_n\,c_{m,n}\frac{\partial f}{\partial c_{m,n}},$$

$$(\xi - x)\frac{\partial f}{\partial(\xi - x)} = -\,wf + \mathrm{S}_m\mathrm{S}_n\,m\,c_{m,n}\frac{\partial f}{\partial c_{m,n}},$$

$$(\eta - y)\frac{\partial f}{\partial(\eta - y)} = -\,wf + \mathrm{S}_m\mathrm{S}_n\,n\,c_{m,n}\frac{\partial f}{\partial c_{m,n}}.$$

The coefficient of the highest power of $\pi - p$ is not only isobaric in x and in y, but is homoiobaric, of weight w, in x and y. Also d_z denotes the degree of that coefficient, d denoting the algebraic degree of f.

We have, as in the case of plane curves, a theory of deduction from a matrix, and a theory of eduction (in the form of a Jacobian function) of both matrices and invariants.

The subject is quite similar at all steps to the investigation for plane curves in this paper, and offers little additional interest except in the solution of the differential equations which the matrices and invariants satisfy. By solution we can verify HALPHEN'S statement in the last paragraph of his thesis (p. 60).

There is an invariant of the second and of the third order, two of the fourth and six of the fifth. There are three matrices, not invariants, of the fourth order, and a system of invariantal coordinates (§ 5 below) can be formed.

The theory of covariants of twisted curves is sufficiently different to merit a separate investigation, but the germ of it already exists in HALPHEN'S paper (quoted above) "On the Invariants of Twisted Curves."

The conditions for covariants of surfaces given above, include those previously obtained by Mr. E. B. ELLIOTT, "On Ternary and n-ary Reciprocants," 'Proc. Math. Soc.,' No. 262, 1886.—*August* 2, 1893.]

§ 2. *On the Conditions of a Covariant Function, and on the modes of Eduction and Development.*

7. *The Form Conditions.*

The equations (A) (B) and (C) determine the form of the covariant functions; we will write them respectively

$$(O_1 - \Omega_1) f = 0 \quad \text{(A)}$$

$$(O_2 - \Omega_2) f = 0 \quad \text{(B)}$$

$$(O_3 - \Omega_3) f = 0 \quad \text{(C)},$$

where

$$O_1 \equiv (\pi - p) \frac{\partial}{\partial (\xi - x)} \qquad \Omega_1 \equiv \mathrm{SS}_n \mathrm{C}_{p-1} y_p y_{n-p+1} \frac{\partial}{\partial y_n}, \text{ \&c.}$$

Expanding the expressions for Ω_1, Ω_2, and Ω_3, we obtain

$${}^{*}\,\Omega_1 = 3y_2^2 \frac{\partial}{\partial y_3} + 10 y_2 y_3 \frac{\partial}{\partial y_4} + (15 y_2 y_4 + 10 y_3^2) \frac{\partial}{\partial y_5} + (21 y_2 y_5 + 35 y_3 y_4) \frac{\partial}{\partial y_6}$$
$$+ (28 y_2 y_6 + 56 y_3 y_5 + 35 y_4^2) \frac{\partial}{\partial y_7} + \text{\&c.}$$

$${}^{*}\,\Omega_2 = 3y_2 \frac{\partial}{\partial y_3} + 8 y_3 \frac{\partial}{\partial y_4} + 15 y_4 \frac{\partial}{\partial y_5} + 24 y_5 \frac{\partial}{\partial y_6} + 35 y_6 \frac{\partial}{\partial y_7} + \text{\&c.}$$

$$\Omega_3 = 6y_2^2 \frac{\partial}{\partial y_4} + 30 y_2 y_3 \frac{\partial}{\partial y_5} + (60 y_2 y_4 + 40 y_3^2) \frac{\partial}{\partial y_6} + (105 y_2 y_5 + 175 y_3 y_4) \frac{\partial}{\partial y_7}$$
$$+ 42 (4 y_2 y_6 + 8 y_3 y_5 + 5 y_4^2) \frac{\partial}{\partial y_8} + \text{\&c.}$$

Relations between the Operators in the Form Conditions.

It is easy to prove that we have the relations

$$\left.\begin{aligned} (O_1 - \Omega_1)(O_2 - \Omega_2) - (O_2 - \Omega_2)(O_1 - \Omega_1) &= O_3 - \Omega_3 \\ (O_2 - \Omega_2)(O_3 - \Omega_3) - (O_3 - \Omega_3)(O_2 - \Omega_2) &= 0 \\ (O_3 - \Omega_3)(O_1 - \Omega_1) - (O_1 - \Omega_1)(O_3 - \Omega_3) &= 0 \end{aligned}\right\} \quad . \quad . \quad (16).$$

---

[*] The operators Ω_2 and Ω_1 are considered most conveniently in the form (25).

$\Omega_2 = 0$ is the partial differential equation which is satisfied by invariants in the theory of forms, and $\Omega_1 = 0$ was established for reciprocants by Professor SYLVESTER, in his lectures "On the Theory of Reciprocants" ('Amer. Jour. Math.,' vol. 8, p. 238).

I have not been able to consult a paper by SOPHUS LIE ('Mathem. Annalen.,' vol. 32), in which I am informed that this operator is applied to kindred purposes.—*August* 2, 1893.

8. *Relation between the operator and d/dx.*

In the functions under consideration, x and y only appear in $\xi - x$, $\pi - p$, and the differential coefficients.

Hence

$$\left.\begin{aligned} \frac{d}{dx} &= -\frac{\partial}{\partial(\xi - x)} - y_2(\xi - x)\frac{\partial}{\partial(\pi - p)} + \mathrm{S}y_{n+1}\frac{\partial}{\partial y_n} \\ &= \quad \mathrm{D}_x + \partial_x, \text{ say,} \\ \text{where} \quad \mathrm{D}_x &= -\frac{\partial}{\partial(\xi - x)} - y_2(\xi - x)\frac{\partial}{\partial(\pi - p)} \\ \text{and} \quad \partial_x &= \quad \mathrm{S}y_{n+1}\frac{\partial}{\partial y_n} \end{aligned}\right\} \quad . \quad . \quad . \quad (17).$$

None of the operators Ω_1, Ω_2, Ω_3, contain $\partial/\partial y_2$. Also we have to bear in mind that the functions do not contain y (except so far as it enters through $\pi - p$), and therefore, in dealing with general terms, we must exclude y_2 when it makes its appearance as if a derivative of y_1.

The division of d/dx into two parts makes the steps somewhat easier, and the final results are

$$\left.\begin{aligned} (\mathrm{O}_1 - \Omega_1)\frac{d}{dx} - \frac{d}{dx}(\mathrm{O}_1 - \Omega_1) &= y_2(w + d + d_x) \\ (\mathrm{O}_2 - \Omega_2)\frac{d}{dx} - \frac{d}{dx}(\mathrm{O}_2 - \Omega_2) &= -(2w - d_x) \\ (\mathrm{O}_3 - \Omega_3)\frac{d}{dx} - \frac{d}{dx}(\mathrm{O}_3 - \Omega_3) &= (\mathrm{O}_1 - \Omega_1) + y_2(\mathrm{O}_2 - \Omega_2) \end{aligned}\right\} \quad . \quad . \quad . \quad (18).$$

Hence if f be a covariant, satisfying the necessary conditions, and if f' stand for df/dx, it follows that

$$\left.\begin{aligned} (\mathrm{O}_1 - \Omega_1)f' &= -y_2(w + d + d_x)f \\ (\mathrm{O}_2 - \Omega_2)f' &= -(2w - d_x)f \\ (\mathrm{O}_3 - \Omega_3)f' &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Now f' satisfies the condition as to weight and degree necessary for being a covariant, simply d and d_x remain unaltered, while w is increased by unity.

Hence if the weight and degree of f are such that

$$w + d + d_x = 0$$
$$2w - d_x = 0,$$

then f' will also be a covariant.

We are thus led to the

9. *Theory of eduction of covariants.*

This theory is clear from what has preceded, but, to give it its simplest form, it is best to return to the preliminary conditions for a covariant.

If ϕ is a covariant, such that $w + d + d_x = 0$ and $2w - d_x = 0$, the multiplying factor is $\mathrm{D}^{d+d_x}\,\nu^{-d}$, and we may write the relation

$$\phi = \mathrm{D}^{d+d_x}\,\nu^{-d}\,\Phi.$$

Then

$$\frac{d\phi}{dx} = \frac{d\mathrm{X}}{dx}\frac{d\phi}{d\mathrm{X}} = \mathrm{D}^{d+d_x}\lambda^{-1}\mu^2\nu^{-d}\frac{d\Phi}{d\mathrm{X}}.$$

Hence $d\phi/dx$ is a covariant in which d and d_x are unaltered and w is increased by unity (3).

Before making use of this theory of eduction it is necessary to find a method of finding some covariant functions from which the eduction must be made, and the method now to be explained will in general supersede this mode of eduction.

10. *Method of development from a matrix.*

We have, however, another method for the formation of a covariant, namely, directly from the differential equations.

Let ${}_{d-m}\Phi$ stand for the collection of terms algebraically homogeneous, and of degree $d - m$, and let ${}_{d-m}\phi_0$, ${}_{d-m}\phi_1$, ${}_{d-m}\phi_2$ stand for the coefficients of terms containing $(\xi - x)^0$, $\xi - x$, $(\xi - x)^2$ among these homogeneous terms. Then, amongst these terms (15A) gives the law of derivation, namely,

$$(q + 1)\,\phi_{q+1} = \Omega_1\phi_q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

a law which is independent of $d - m$, and for all values of $d - m$, $\Omega_1\,{}_{d-m}\phi_{d-m} = 0$.

Hence, if we know ${}_d\phi_0$, the coefficient of the highest power of $\pi - p$, we find all the terms of the d^{th} degree.

11. *Modes of derivation of homogeneous terms from those of a higher degree.*

The conditions (15B) and (15C) give us two modes of derivation, which imply the existence of certain relations and conditions.

We have from (15B) and (15C) respectively,

$$\begin{aligned}(\xi - x)\,\{{}_{d-1}\Phi + 2\,{}_{d-2}\Phi + \dots \quad &+ (m+1)\,{}_{d-m-1}\Phi + \dots\} \\ + \Omega_2\,\{{}_d\Phi + {}_{d-1}\Phi + \dots \quad &+ {}_{d-m}\Phi + \dots\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (21),\end{aligned}$$

$$\begin{aligned}(\pi - p)\,\{{}_{d-1}\Phi + 2\,{}_{d-2}\Phi + \dots \quad &+ (m+1)\,{}_{d-m-1}\Phi + \dots\} \\ + \Omega_3\,\{{}_d\Phi + {}_{d-1}\Phi + \dots \quad &+ {}_{d-m}\Phi + \dots\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (22).\end{aligned}$$

Or,

$$\begin{aligned} (m+1)(\xi - x)\,{}_{d-m-1}\Phi + \Omega_2\,{}_{d-m}\Phi &= 0, \\ (m+1)(\pi - p)\,{}_{d-m-1}\Phi + \Omega_3\,{}_{d-m}\Phi &= 0, \end{aligned}$$

and from these we again conclude

$$\left.\begin{aligned} (m+1)\,{}_{d-m-1}\phi_q + \Omega_2\,({}_{d-m}\phi_{q+1}) &= 0 \\ (m+1)\,{}_{d-m-1}\phi_q + \Omega_3\,({}_{d-m}\phi_q) &= 0 \end{aligned}\right\} \quad \ldots\ldots \quad (23).$$

From (20) we deduce that

$$\{\Omega_2\Omega_1 - (q+1)\,\Omega_3\}\,\phi_q = 0 \quad \ldots\ldots\ldots \quad (24).$$

That this is an identical relation among the coefficients appears from considering the operator.

$$\{\Omega_2\Omega_1 - (q+1)\,\Omega_3\}\,\Omega_1{}^q$$

which

$$\begin{aligned} &= \{\Omega_1\Omega_2 - q\,\Omega_3\}\,\Omega_1{}^q \\ &= \{\Omega_1\Omega_2\Omega_1 - q\,\Omega_3\Omega_1\}\,\Omega_1{}^{q-1} \\ &= \Omega_1\,\{\Omega_2\Omega_1 - q\,\Omega_3\}\,\Omega_1{}^{q-1} \\ &= \Omega_1\,\{\Omega_1\Omega_2 - (q-1)\,\Omega_3\}\,\Omega_1{}^{q-1} \\ &\quad \&\text{c.}, \end{aligned}$$

and ultimately

$$= \Omega_1{}^{q+1}\Omega_2.$$

But

$$\Omega_2\phi_0 = 0,$$

hence

$$\{\Omega_2\Omega_1 - (q+1)\,\Omega_3\}\,\phi_q = 0.$$

There is then no further condition to be satisfied among the coefficients in the expansion. The importance of this will be seen later in the development of covariants.

12. The order of derivation is symbolised by a chart of the coefficients, with arrows showing the direction of derivation. Thus:—

$$\begin{tikzcd}
{}_{d}\phi_0 \arrow[rr] \arrow[dr] & & {}_{d}\phi_1 \arrow[rr] \arrow[dl] \arrow[dr] & & {}_{d}\phi_2 \arrow[rr] \arrow[dl] \arrow[dr] & & {}_{d}\phi_3 \arrow[dl] & \cdot\ \cdot\ \cdot \\
& {}_{d-1}\phi_0 \arrow[rr] \arrow[dr] & & {}_{d-1}\phi_1 \arrow[rr] \arrow[dl] \arrow[dr] & & {}_{d-1}\phi_2 \arrow[dl] \arrow[dr] & & \cdot\ \cdot\ \cdot \\
& & {}_{d-2}\phi_0 \arrow[rr] & & {}_{d-2}\phi_1 \arrow[rr] & & {}_{d-2}\phi_2 & \cdot\ \cdot \\
& & & & \cdot & & \cdot &
\end{tikzcd}$$

where

$\longrightarrow$ denotes deduction by the operator Ω_1,

$\swarrow$,, ,, ,, ,, Ω_2,

$\searrow$,, ,, ,, ,, Ω_3,

in which we can easily trace the relations (16)

$$\Omega_2\Omega_1 - \Omega_1\Omega_2 = \Omega_3,$$
$$\Omega_2\Omega_3 = \Omega_3\Omega_2,$$
$$\Omega_1\Omega_3 = \Omega_3\Omega_1.$$

According to this mode of procedure ${}_d\phi_0$ is the matrix from which the other coefficients in the covariant are developed. It is necessary that ${}_d\phi_0$ should be a homogeneous isobaric function of the differential coefficients satisfying the differential equation $\Omega_2 f = 0$. It is also essential that it shall not be an invariant as then no development ensues. Taking the solution of $\Omega_2 f = 0$, including only differential coefficients up to the fourth order, we obtain ${}_2\phi_0 = 3y_2y_4 - 4y_3^2$, and from it the general covariant of conics may be developed.

13. *General solution of* $\Omega_2 f = 0$, *and theory of eduction of matrices.*

Confining ourselves to the coefficients in the covariant functions, the equations (18) may be written

$$\left.\begin{aligned}\Omega_1\frac{d}{dx} - \frac{d}{dx}\Omega_1 &= -y_2(w + d_x)\\ \Omega_2\frac{d}{dx} - \frac{d}{dx}\Omega_2 &= (2w - d_x)\\ \Omega_3\frac{d}{dx} - \frac{d}{dx}\Omega_3 &= \Omega_1 + y_2\Omega_2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18\text{A}).$$

Hence, if ϕ is a homogeneous isobaric function of the differential coefficients such that $\Omega_2\phi = 0$, then

$$\Omega_2\frac{d\phi}{dx} = (2w - d_x)\phi,$$

and if the weight and degree of ϕ are such that $2w = d_x$, then $d\phi/dx$ also satisfies the same differential equation, and it is homogeneous and isobaric.

The solutions of $\Omega_2 f = 0$ up to the fourth order are

$$y_2 \text{ and } 3y_2y_4 - 4y_3^2\,;$$

from these we form, so as to satisfy $2w - d_x = 0$,

$$\phi = (3y_2y_4 - 4y_3^2)^3\,y_2^{-10},$$

and obtain from it by differentiation the educt of the fifth order

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3,$$

which is the differential invariant of conics, and does not produce any covariant by development.

By this process of eduction, a solution can be found for each order, but it is possible to find a simpler series of solutions of $\Omega_2 f = 0$ than those thus obtained.

14. *Extension of the theory of eduction of matrices.*

If ϕ is a matrix, and if we write ϕ_1, ϕ_2, &c. for $d\phi/dx$, $d^2\phi/dx^2$ &c., it is easily seen that

$$\begin{aligned}\Omega_2\phi_2 &= \frac{d}{dx}\left(\Omega_2\frac{d\phi}{dx}\right) + (2w - d_x + 2)\,\phi_1,\\ &= 2\,(2w - d_x + 1)\,\phi_1.\end{aligned}$$

Similarly,

$$\begin{aligned}\Omega_2\phi_3 &= 3\,(2w - d_x + 2)\,\phi_2, \text{ \&c.},\\ \Omega_2\phi_n &= n\,(2w - d_x + n - 1)\,\phi_{n-1},\end{aligned}$$

where w and d_x are the order and degree of ϕ.

Hence

$$(2w - d_x)\,\phi\phi_2 - (2w - d_x + 1)\,\phi_1^2$$

and

$$(2w - d_x)^2\,\phi^2\phi_3 - 3\,(2w - d_x)(2w - d_x + 2)\,\phi\phi_1\phi_2 + 2\,(2w - d_x + 1)(2w - d_x + 2)\,\phi_1^3$$

are matrices, and are of the form of the matrix and differential invariant of conics.

Generally, if ϕ and ψ are two matrices, and if α and β are the respective values of $2w - d_x$ for each,

$$\beta\phi\psi_1 - \alpha\phi_1\psi$$

is a matrix.

Convenient forms of the differential equations and of the irreducible matrices.

In obtaining the differential equations of condition (15), and in considering the method of eduction by differentiation, there has been a convenience in retaining the differential coefficients y_2, y_3, &c., but the forms of the equations $\Omega f = 0$, and of their solutions, are simplified by replacing them by a_2, a_3, &c., where

$$a_n = y_n/n\,!.$$

We thus obtain (15)

$$\left.\begin{aligned}
\Omega_1 &= 2a_2^2\frac{\partial}{\partial a_3} + 5a_2a_3\frac{\partial}{\partial a_4} + (6a_2a_4 + 3a_3^2)\frac{\partial}{\partial a_5} + (7a_2a_5 + 7a_3a_4)\frac{\partial}{\partial a_6} \\
&\qquad + (8a_2a_6 + 8a_3a_5 + 4a_4^2)\frac{\partial}{\partial a_7} + \&c. \\
\Omega_2 &= a_2\frac{\partial}{\partial a_3} + 2a_3\frac{\partial}{\partial a_4} + 3a_4\frac{\partial}{\partial a_5} + \&c. \\
\Omega_3 &= a_2^2\frac{\partial}{\partial a_4} + 3a_2a_3\frac{\partial}{\partial a_5} + (4a_2a_4 + 2a_3^2)\frac{\partial}{\partial a_6} + (5a_2a_5 + 5a_3a_4)\frac{\partial}{\partial a_7} \\
&\qquad + (6a_2a_6 + 6a_3a_5 + 3a_4^2)\frac{\partial}{\partial a_8} + \&c.
\end{aligned}\right\} \quad . \quad (25).$$

The irreducible matrices of even order are easily seen to be

$$\left.\begin{aligned}
u_2 &= a_2 \\
u_4 &= a_2a_4 - a_3^2 \\
u_6 &= a_2a_6 - 4a_3a_5 + 3a_4^2 \\
u_8 &= a_2a_8 - 6a_3a_7 + 15a_4a_6 - 10a_5^2
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (26),$$

and, generally, when n is even,

$$u_n = a_2a_n - (n-2)\,a_3a_{n-1} + \frac{(n-2)\,(n-3)}{1\,.\,2}\,a_4a_{n-2} - \&c.,$$

the last coefficient being half of the value as appearing from this series.

The matrices of odd orders are found from these by the last form of the process of eduction and are

$$\left.\begin{aligned}
u_5 &= a_2^2a_5 - 3a_2a_3a_4 + 2a_3^3 \\
u_7 &= a_2^2a_7 - 5a_2a_3a_6 + 2a_2a_4a_5 + 8a_3^2a_5 - 6a_3a_4^2 \\
u_9 &= a_2^2a_9 - 7a_2a_3a_8 + 9a_2a_4a_7 - 5a_2a_5a_6 + 12a_3^2a_7 - 30a_3a_4a_6 + 20a_3a_5^2
\end{aligned}\right\} \quad . \quad (27).$$

From these it follows that

$$\left.\begin{aligned}
\Omega_3u_2 &= 0 \\
\Omega_3u_4 &= u_2^3 \\
\Omega_3u_5 &= 0 \\
\Omega_3u_6 &= 10u_2u_4 \\
\Omega_3u_7 &= 7u_2u_5 \\
\Omega_3u_8 &= 21u_2u_6 \\
\Omega_3u_9 &= 16u_2u_7 \\
\&c.&
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28),$$

and

$$\left.\begin{aligned} \Omega_1 u_2 &= 0 \\ \Omega_1 u_4 &= u_2^3 a_3 \\ \Omega_1 u_5 &= 0 \\ \Omega_1 u_6 &= - u_5 + 10 u_4 a_3 \\ \Omega_1 u_7 &= - 2u_2^2 u_6 + 10 u_4^2 + 7 u_5 a_3 \\ \Omega_1 u_8 &= - 3u_7 + 21 u_6 a_3 \\ u_2^2 \Omega_1 u_9 &= - 4u_2^4 u_8 + 52 u_2^2 u_4 u_6 - 30 (u_5^2 + 4u_4^3) + 16 u_2^2 u_7 a_3 \end{aligned}\right\} \quad . \quad . \quad (29).$$

15. *Reduction of the matrices to functions of differential invariants, and two fundamental matrices of orders* 4 *and* 6.

On the reduction which is now introduced depends the possibility of making the subject of this paper an instrument of research into the character of the higher curves.

For every order higher than the third, there is either a possible matrix or a differential invariant, and for every order higher than the sixth there is a differential invariant. Of the third order there is neither, and of the fourth and sixth orders there is no differential invariant. Every function of differential coefficients can be expressed as a function of a_3, matrices of orders 4 and 6, and differential invariants, and if the function is itself a matrix it will not contain a_3.

As a first step, I replace the irreducible matrices of the sixth and higher orders, by matrices of homogeneous covariants, that is, by functions of the irreducible matrices which satisfy $\Omega_3 f = 0$.

As far as the 9th order these are the irreducible solutions of

$$\frac{du_2}{0} = \frac{du_4}{u_2^2} = \frac{du_5}{0} = \frac{du_6}{10u_4} = \frac{du_7}{7u_5} = \frac{du_8}{21u_6} = \frac{du_9}{16u_7},$$

and are

$$\left.\begin{aligned} L_6 &= u_2^2 u_6 - 5u_4^2 \\ L_7 &= u_2^2 u_7 - 7u_4 u_5 \\ L_8 &= u_2^4 u_8 - 21 u_2^2 u_4 u_6 + 70 u_4^3 \\ L_9 &= u_2^4 u_9 - 16 u_2^2 u_4 u_7 - 30 u_2^2 u_5 u_6 + 206 u_4^2 u_5 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (30).$$

From these again we find

$$\begin{aligned} \Omega_1 L_6 &= - u_2^2 u_5 \\ \Omega_1 L_7 &= - 2u_2^2 L_6 \\ \Omega_1 L_8 &= - 3u_2^2 L_7 \\ \Omega_1 L_9 &= - 4u_2^2 L_8 . \end{aligned}$$

From which we find the differential invariants of orders 7 and 8 conveniently in the forms

$$\left.\begin{aligned} U_7 &= u_5 L_7 - L_6^2 \\ U_8 &= u_5^2 L_8 - 3u_5 L_6 L_7 + 2L_6^3 \\ U_9 &= u_5 L_9 - 4L_6 L_8 + 3L_7^2 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (31),$$

which present the same forms as the irreducible matrices of order 4, 5, and 6.

16. *The differential coefficients of the irreducible matrices, and of the quantities by which they are replaced.*

In this section of the paper the relations between the matrices, &c., are being investigated for the purpose of use in the next section.

The differential coefficients of the irreducible matrices of even order are readily written down, and those of the matrices of odd orders with rather more difficulty. Thus

$$\left.\begin{aligned} u_2 \frac{du_4}{dx} &= 5u_5 + 10u_4 a_3 \\ u_2 \frac{du_6}{dx} &= 7u_7 + 14u_6 a_3 \\ u_2 \frac{du_8}{dx} &= 9u_9 + 18u_8 a_3 \\ &\text{and} \\ u_2 \frac{du_5}{dx} &= 6L_6 + 15u_5 a_3 \\ u_2^3 \frac{du_7}{dx} &= 8L_8 + 40u_4 L_6 + 90u_5^2 + 19u_2^2 u_7 a_3. \\ &\text{Hence} \\ u_2 u_5 \frac{dL_6}{dx} &= 7U_7 + 7L_6^2 - u_4 u_5^2 + 20u_5 L_6 a_3 \\ u_2 u_5 \frac{dU_7}{dx} &= 8U_8 + 16U_7 L_6 + 55u_5^4 + 40u_5 U_7 a_3 \\ u_2 u_5 \frac{dU_8}{dx} &= 9u_5^3 L_9 - 21U_7^2 - 12U_8 L_6 - 54U_7 L_6^2 - 9L_6^4 + 60u_5 U_8 a_3 \end{aligned}\right\} \quad . \quad (32).$$

17. *Expressions for a_2, a_3, a_4, &c., in terms of u_4, L_6, differential invariants and a_3.*

These expressions are found consecutively by the aid of the formulæ which have just been established. Thus,

$$\left.\begin{aligned}
a_2 &= u_2 \\
a_3 &= a_3 \\
a_4 &= \frac{u_4 + a_3^2}{u_2} \\
a_5 &= \frac{u_5 + 3u_4 a_3 + a_3^3}{u_2^2} \\
a_6 &= \frac{\mathrm{L}_6 + 2u_4^2 + 4u_5 a_3 + 6u_4 a_3^2 + a_3^4}{u_2^3} \\
a_7 &= \frac{\mathrm{U}_7 + \mathrm{L}_6^2 + 5u_4 u_5^2 + 5u_5(\mathrm{L}_6 + 2u_4^2)\,a_3 + 10u_5^2 a_3^2 + 10u_4 u_5 a_3^3 + a_3^5}{u_2^4 u_5} \\
a_8 &= \frac{\mathrm{U}_8 + 10u_5^4 + 3\mathrm{U}_7\mathrm{L}_6 + \mathrm{L}^3_6 + 6u_4 u_5^2\mathrm{L}_6 + 5u_4^3 u_5^2 + 6u_5(\mathrm{U}_7 + \mathrm{L}_6^2 + 5u_4 u_5^2)\,a_3 + \&\mathrm{c}.}{u_2^5 u_5^2} \\
a_9 &= \frac{u_5^2\mathrm{U}_9 - 3\mathrm{U}_7^2 + \&\mathrm{c}.}{u_2^6 u_5^3}
\end{aligned}\right\} \quad (33).$$

In these formulæ, and in their applications, it must be remembered that they imply that u_2 and u_5 are not zero at the point to which they refer.

§ 3. *Correlative Forms.*

18. *The deduction of the equation to the reciprocal of a covariant from the equation to the original covariant.*

[The principle of duality, as applied to differential invariants, is explained by HALPHEN (p. 56) in his thesis. In the case of covariants, the relation is still more interesting.

The general investigation will come later, but let us begin by considering the connection in its most elementary form. Let x, y, be the coordinates on a covariant in either point- or line-coordinates, and X, Y, the corresponding coordinates on the reciprocal in line- or point-coordinates. We express the correlative transformation by

$$\left.\begin{aligned} \mathrm{X} &= y_1 \\ \mathrm{Y} &= xy_1 - y \end{aligned}\right\} \quad \text{and inversely} \quad \left\{\begin{aligned} x &= \mathrm{Y}_1 \\ y &= \mathrm{XY}_1 - \mathrm{Y} \end{aligned}\right.$$

y_1 and Y_1 being understood to stand for dy/dx and $d\mathrm{Y}/d\mathrm{X}$. Then we easily find

$$\mathrm{Y}_2 = \frac{1}{y_2}, \quad \mathrm{Y}_3 = -\frac{y_3}{y_2^3}, \quad \mathrm{Y}_4 = -\frac{y_4}{y_2^4} + \frac{3y_3^2}{y_2^5}, \quad \text{and so on,}$$

with the inverse relationships.

These equations enable us to express any function of Y_2, Y_3, Y_4, &c., as an equivalent function of y_2, y_3, y_4, &c., and HALPHEN has shown the relationship between invariant functions.

Our concern now is with covariants and especially with matrices.

Obviously, the conditions to which a function is subject, in order that it may be a covariant function, are of the same form, whether the coordinates are point- or line-coordinates, and therefore the matrix of the reciprocal of a covariant, regarded as a function of Y_2, Y_3, &c., is a solution of $\Omega_2 F = 0$. It is proposed, in the first place, to deduce the equation to the reciprocal from the equation to the original covariant, and, found in this way, the coefficient which we have called the matrix will be a function of y_2, y_3, &c., and as a function of these quantities it will not be a solution of $\Omega_2 f = 0$.

To find the equation which it does satisfy, we will simplify the writing by taking the point x, y as origin, so that we may write π for $\pi - p$ and ξ for $\xi - x$. The equation to the covariant is a rational integral function in π and ξ.

Let γ and α be the corresponding coordinates on the reciprocal curve with the relation

$$\alpha\xi - \pi - \gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

where the third letters which are usually inserted to make the equation homogeneous are omitted as unnecessary and, for the purpose, undesirable.

Following the usual course (SALMON, 'Higher Plane Curves,' 2nd ed., p. 73), the equation to the covariant is now made homogeneous by using this equation, and the discriminant of the resulting equation, considered as a binary quantic in π and ξ, equated to zero, is the required equation to the reciprocal.

It is not necessary that the whole of this process should be performed, for all that is necessary is to obtain the coefficient of the highest power of γ in the resulting equation. We obtain it by putting $\alpha = 0$, and hence *the coefficient of the highest power of* γ *is the discriminant of the highest group of homogeneous term in the original equation in* π *and* ξ, *treated as a binary quantic.*

This coefficient is, of course, found as a function of y_2, y_3, . . . , and when replaced by the corresponding value in terms of Y_2, Y_3, it will be the matrix of the reciprocal and a solution of $\Omega_2 F = 0$.

From the mode of formation of the coefficient of the highest power of γ, it appears that, as a function of y_2, y_3, . . . , it is an invariant for changes of Cartesian coordinates, and therefore, as has been remarked in finding the condition resulting from the coefficient B_1 in the original homographic transformation (12), it will satisfy the equation $\Omega_1 f = 0$.

Before continuing the more general consideration of this relation we will illustrate this on the general osculating conic and its reciprocal.

The terms of the highest order in the covariant conic are

$$(a_2 a_4 - a_3^2)\,\pi^2 + a_2^2 a_3 \pi\xi + a_2^4 \xi^2,$$

and, therefore, the coefficient of γ^2 in the reciprocal will be, omitting a power of a_2,

$$4a_2a_4 - 5a_3^2,$$

and this is seen to satisfy $\Omega_1 f = 0$ (25).

On introduction of A_2, A_3, A_4 to replace Y_2, Y_3, Y_4, as a_2, a_3, a_4 replace y_2, y_3, and y_4, and omitting a power of A_2, we obtain

$$A_2A_4 - A_3^2$$

for the matrix of the reciprocal conic as a function of A_2, A_3, A_4, in which form it is, of course, a solution of $\Omega_2 F = 0$.—*Rewritten August* 2, 1893.]

19. *Dual relation between the operators Ω_1 and Ω_2.*

The methods of solution of $\Omega_1 f = 0$ and $\Omega_2 f = 0$, are alike, inasmuch as each depends upon a method of eduction starting from the solutions of the second and fourth orders. The general proposition is now stated, that if u be a solution of either $\Omega_1 f = 0$ or $\Omega_2 f = 0$, in which y_2, y_3, &c., or, what is the same thing, a_2, a_3, &c., are the arguments, and if in consequence of the substitution for these quantities of Y_2, Y_3, &c., or of A_2, A_3, &c., u becomes U, then U is a solution of the other corresponding equation, that is, of $\Omega_2 F = 0$ or $\Omega_1 F = 0$.

As far as the first three solutions of $\Omega_1 f = 0$ are concerned, it is readily verified that

$$u_2, \qquad u_4, \qquad \text{and} \qquad u_5,$$

become

$$U_2, \qquad 4A_2A_4 - 5A_3^2, \qquad \text{and} \qquad U_5,$$

divided respectively by U_2^2, $-U_2^6$, and $-U_2^9$ and numerical factors, where the index of the power of U_2 is the weight of the function which it affects.

If θ be a function of a_2, a_3, &c., of weight w and degree d_x, on substitution it will become Θ/U_2^w, where Θ has the weight and degree of θ. Writing W and D_x for the weight and degree of Θ/U_2^w we obtain

$$W = -w, \qquad D_x = d_x - w,$$

so that

$$W + D_x = -(2w - d_x) \quad \text{and} \quad -(2W - D_x) = w + d_x \quad . \quad . \quad . \quad (35).$$

Bearing in mind the conditions (18A) for the eduction of solutions of $\Omega_1 f = 0$ and $\Omega_2 f = 0$, it follows that processes which educe solutions of $\Omega_2 f = 0$ from functions of the character of θ, will educe solutions of $\Omega_1 F = 0$ from functions of the character of Θ/U_2^w, and *vice versâ*.

As a consequence, in making a change from point-coordinates to the correlated line coordinates, or *vice versâ*, the operators Ω_1 and Ω_2 are interchanged. Or expressed in terms of a_2, a_3, &c., the coefficient of the highest power of γ in a contravariant is a solution of $\Omega_1 f = 0$.

The proof just given is connected with the theory of eduction, but it is perhaps more simple to notice that the relation

$$\alpha\xi - \pi - \gamma = 0$$

must hold not only for the original coordinates, but also for the coordinates in the general homographic transformations, and hence to deduce the dual relationship from the essential principles of this paper.

That is, if we make infinitesimal homographic transformation for both ξ, π, and α, γ, indicated by

$$\frac{\xi}{\xi' + B_1\pi'} = \frac{\pi}{\pi'} = \frac{1}{A\xi' + B\pi' + 1},$$

$$\frac{\alpha}{\alpha' + D_1\gamma'} = \frac{\gamma}{\gamma'} = \frac{1}{C\alpha' + D\gamma' + 1},$$

as in Art. 2,
then, in consequence of

$$\alpha\xi - \pi - \gamma = 0,$$

$$B_1\pi'\alpha' + D_1\xi'\gamma' = \pi'(C\alpha' + D\gamma') + \gamma'(A\xi' + B\pi').$$

and

$$B_1 = C, \qquad A = D_1, \qquad D + B = 0.$$

Or A is replaced by (C or by) B_1, and B_1 by (D_1 or by) A, while B is replaced by (D or by) $-$ B.

Hence the conditions derived from A and B_1 are interchanged, while those derived from B remain unaltered (Art. 6). That is Ω_1 and Ω_2 are interchanged, and Ω_3 is unaltered.

20. *Relations of the differential invariants, and the solutions of* $\Omega_1 f = 0$ *and* $\Omega_2 f = 0$ *to the ordinary invariants of a curve.*

Differential invariants appear in the processes of eduction, that is ultimately as the result of differentiation, in three ways. Firstly, in a series of differential invariants only; secondly, from a series of solutions of $\Omega_1 f = 0$, *i.e.*, a series of differential invariants for change of coordinates or semi-invariants; and thirdly, from the series which we have called matrices, which are semi-invariants in the correlated system of coordinates.

The differential equation to a curve involves differential invariants only. If we find the series of first integrals (in the next section I illustrate, on the cubic, the general method of finding this series), we may form from among them the functions of differential invariants, of semi-invariants, and of matrices which are constant along the curve. These functions may be properly called (each in its proper sphere, as homographic or Cartesian) differential expressions for the invariants of the curve.

It is perhaps worthy of remark that in showing the mode of integrating the differential equation of conics, Mr. ELLIOTT[*] does, in fact, illustrate these methods of procedure.

§ 4. *Application of Results to the Cubic.*

21. *The differential equation of the general cubic.*

It has been previously shown that the order of this differential equation will be 9, the degree 10, and the weight 35. It is also a function of u_5, U_7, U_8, and U_9, since it is easily seen that u_2 cannot explicitly enter the equation.

Written as a determinant, the equation is

$$\begin{vmatrix} a_9, & a_8, & a_7, & 2a_3a_6+2a_4a_5, & a_4^2+2a_3a_5+2a_2a_6, & a_3^3+6a_2a_3a_4+3a_2^2a_5 \\ a_8, & a_7, & a_6, & a_4^2+2a_3a_5, & 2a_2a_5+2a_3a_4, & 3a_2^2a_4+3a_2a_3^2 \\ a_7, & a_6, & a_5, & 2a_3a_4, & a_3^2+2a_2a_4, & 3a_2^2a_3 \\ a_6, & a_5, & a_4, & a_3^2, & a_2a_3, & a_2^3 \\ a_5, & a_4, & a_3, & 0, & a_2^2, & 0 \\ a_4, & a_3, & a_2, & -a_2^2, & 0 & 0 \end{vmatrix} = 0.$$

From this we see that a_8 and a_9 enter only in the form $a_7a_9 - a_8^2$.

Hence we have the terms $u_5^2U_7U_9 - U_8^2 + \ldots$ Since the degree of these terms is 24, and weight 72, the differential invariant when complete must be divisible by $u_2^5u_5^3$.

The differential equation therefore takes the form

$$u_5^2U_7U_9 - (U_8^2 + 4U_7^3) + Eu_5^4U_8 + Tu_5^8 = 0.$$

Determining E and T, either by the condition of containing u_2^5 as further factors or by comparison with the determinant, we obtain

$$u_5^2U_7U_9 - U_8^2 - 4U_7^3 - 15U_5^4U_8 - 56U_5^8 = 0,$$

or

$$u_5^2U_7U_9 - V_8^2 - 4U_7^3 + u_5^4V_8 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

where

$$V_8 = U_8 + 8u_5^4.$$

22. *The first integrals of the differential equation of the general cubic.*

From the equation to the general cubic, found as a differential covariant, it is possible to find a complete set of the first integrals of the differential equation.

For the equation to the cubic being written in the form

$$\begin{aligned} b\,(\eta - y_1\,\xi - y + y_1x)^3 + 3b_1\,(\eta - y_1\,\xi - y + y_1x)^2\,(\xi - x) \\ + 3a_2\,(\eta - y_1\,\xi - y + y_1x)\,(\xi - x)^2 + a\,(\xi - x)^3 \\ + 3b_3\,(\eta - y_1\,\xi - y + y_1x)^2 + 6m\,(\eta - y_1\,\xi - y + y_1x)\,(\xi - x) \\ + 3a_3\,(\xi - x)^2 + 3c_2\,(\eta - y_1\,\xi - y + y_1x) = 0, \end{aligned}$$

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[*] 'Messenger of Mathematics,' vol. 19, p. 5, 1889.

and expressing it in terms of powers of ξ and η, we get

$$\begin{aligned} & b\eta^3 \\ & -3\,(by_1 - b_1)\,\eta^2\xi \\ & +3\,(by_1^2 - 2b_1y_1 + a_2)\,\eta\xi^2 + \&\text{c.} \\ & -3\,\{b\,(y - y_1x) + b_1x - b_3\}\,\eta^2 + \&\text{c.} = 0\,; \end{aligned}$$

the coefficients being easily written down.

Now, whatever be the point $(x.y)$ taken upon the curve, we obtain the equation to the curve in an identical form, and therefore the ratios of corresponding coefficients in any two such equations are equivalent.

This being the case, we obtain nine first integrals of the differential equation to the cubic, to which we may give the forms

$$\begin{aligned} & y_1 - \frac{b_1}{b} = \text{constant}, \\ & y_1^2 - 2\,\frac{b_1}{b} + \frac{a_2}{b} = \text{constant}, \\ & y - y_1x + \frac{b_1}{b}\,x - \frac{b_3}{b} = \text{constant}, \\ & \quad \&\text{c.} \end{aligned}$$

On account of the obvious character of these expressions, I do not write them down, nor shall I perform the simple processes whereby they are simplified, but merely write down the final forms in which I propose to deal with them.

For this purpose, write

$$\left.\begin{aligned} & \frac{a_2}{b} - \frac{b_1^2}{b^2} = h_1 \\ & \frac{m}{b} - \frac{b_1b_3}{b^2} = h_2 \\ & \frac{c_2}{b} - \frac{b_3^2}{b^2} = h_3 \\ & \frac{a}{b} - 3\,\frac{a_2b_1}{b^2} + 2\,\frac{b_1^3}{b^3} = k_1 \\ & \frac{a_3}{b} - \frac{a_2b_3}{b^2} - 2\,\frac{b_1}{b}\left(\frac{m}{b} - \frac{b_1b_3}{b^2}\right) = k_2 \\ & \frac{2mb_3}{b^2} + \frac{b_1c_2}{b^2} - 2\,\frac{b_1b_3^2}{b^3} = k_3 \\ & 3\,\frac{c_2b_3}{b_2} - 2\,\frac{b_3^3}{b^3} = k_4. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (37).$$

Then the nine integrals take the form

$$\left.\begin{aligned} &y_1 - \frac{b_1}{b} = \text{constant} \\ &y - y_1 x + \frac{b_1}{b}x - \frac{b_3}{b} = \text{constant} \\ &h_1 x - h_2 = \text{constant}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (38),$$

$$\left.\begin{aligned} h_1 &= \text{constant} \\ k_1 &= \text{constant} \\ h_2k_1 - h_1k_2 &= \text{constant} \\ h_1h_3 - h_2^2 &= \text{constant} \\ k_1k_3 + k_2^2 &= \text{constant} \\ k_1^2k_4 - 3k_1k_2k_3 - 2k_3^2 &= \text{constant} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (39),$$

of which the last six do not contain x, y, or y_1.

To select from among these six first integrals those which are functions of the matrices only.

It is readily proved that

$$\begin{aligned} \Omega_2h_1 &= -2h_2, & \Omega_2k_1 &= -3k_2, \\ \Omega_2h_2 &= -h_3, & \Omega_2k_2 &= 2k_3, \\ \Omega_3h_3 &= 0, & \Omega_2k_3 &= -k_4, \\ & & \Omega_2k_4 &= 0, \end{aligned}$$

and forming the functional solutions of

$$\frac{dh_1}{2h_2} = \frac{dh_2}{h_3} = \frac{dh_3}{0} = \frac{dk_1}{3k_2} = \frac{dk_2}{-2k_3} = \frac{dk_3}{k_4} = \frac{dk_4}{0} \quad \ldots\ldots \quad (40)$$

we get

$$\left.\begin{aligned} &h_3 \\ &k_4 \\ &h_1k_4 - h_3k_3 \\ &h_1h_3 - h_2^2 \\ &k_4k_2 + k_3^2 \\ &k_4^2k_1 - 3k_4k_3k_2 - 2k_2^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (41),$$

forms which are at once recognized as being obtained from those of the last six integrals (39), by interchange of the subscripts 1 and 3 of h, and of 1 and 4, 2 and 3 of k.

Hence the functions involved in the first integrals are the solutions of a set of differential equations analogous to (40), and it is the common solutions of these two sets of equations treated as simultaneous that we are seeking. It is easy to prove that the number of such simultaneous solutions is four, and they can easily be constructed as follows :—

$$\left.\begin{aligned} \mathrm{P}_1 &= h_1h_3 - h_2^2 \\ \mathrm{P}_2 &= h_1(k_2k_4 + k_3^2) + h_2(k_2k_3 - k_1k_4) + h_3(k_1k_3 + k_2^2) \\ \mathrm{P}_3 &= k_1^2k_4^2 - 6k_1k_2k_3k_4 - 4k_1k_3^3 - 4k_2^3k_4 - 3k_2^2k_3^2 \\ \mathrm{P}_4 &= h_1(h_1k_4 - 2h_2k_3 - h_3k_2)^2 + 2h_2(h_1k_4 - 2h_2k_3 - h_3k_2)(h_3k_1 - 2h_2k_2 - h_1k_3) \\ &\qquad + h_3(h_3k_1 - 2h_2k_2 - h_1k_3)^2 \end{aligned}\right\} (42).$$

Of these P_1 can be shown to be Matrix of Hessian/(Matrix of General Cubic)^3.

23. *To find the first integral of the differential equation to the general cubic which is a differential invariant.*

For this purpose we are to find a function of the P's which is a solution of $\Omega_1 f = 0$. With rather more difficulty than in the case of Ω_2, we find

$$\Omega_1 h_1 = k_1 - 4\frac{b_1}{b}h,$$

$$\Omega_1 h_2 = k_2 - 3\frac{b_1}{b}h_2 - \frac{b_3}{b}h_1,$$

$$\Omega_1 h_3 = -k_3 - 2\frac{b_1}{b}h_3 - 2\frac{b_3}{b}h_2,$$

$$\Omega_1 k_1 = -6h_1^2 - 6\frac{b_1}{b}k_1,$$

$$\Omega_1 k_2 = -6h_1h_2 - 5\frac{b_1}{b}k_2 - \frac{b_3}{b}k_1,$$

$$\Omega_1 k_3 = 2h_1h_3 + 4h_2^2 - 4\frac{b_1}{b}k_3 + 2\frac{b_3}{b}k_2,$$

$$\Omega_1 k_4 = 6h_2h_3 - 3\frac{b_1}{b}k_4 - 3\frac{b_3}{b}k_3.$$

The problem is now to find functions of the P's, which satisfy simultaneously

$$\frac{dh_1}{k_1} = \frac{dh_2}{k_2} = \frac{dh_3}{-k_3} = \frac{dk_1}{-6h_1^2} = \frac{dk_2}{-6h_1h_2} = \frac{dk_3}{2h_1h_3 + 4h_2^2} = \frac{dk_4}{6h_2h_3},$$

and

$$\frac{dh_1}{4h_1} = \frac{dh_2}{3h_2} = \frac{dh_3}{2h_3} = \frac{dk_1}{6k_1} = \frac{dk_2}{5k_2} = \frac{dk_3}{4k_3} = \frac{dk_4}{3k_4},$$

the third equation being satisfied by each of the six first integrals.

The first of these equations is satisfied by

$$\left.\begin{aligned}&P_1^2 - P_2\\ \text{and}\quad &2P_4 + 3P_3 - 10P_1P_2 + 14P_1^3\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (43).$$

Hence

$$(P_1^2 - P_2)^3/(2P_4 + 3P_3 - 10P_1P_2 + 14P_1^3)^2 \quad . \; . \; . \; . \; . \quad (44),$$

is the function of differential invariants which gives a first integral of the differential equation, or in other words, is the absolute invariant of the cubic.

24. *The general form of the matrix of a cubic.*

In all cases the matrix is a function of u_4, L_6, and the differential invariants. In the most general case the matrix of the cubic will contain u_4^3, but if we take the cubic to touch the standard curve (which is actually to be the curve itself), the matrix would contain only u_4^2.

The coefficient of u_4^2 will also be simply a differential invariant, and, from the nature of the operators Ω_1 and Ω_3, the complete form is found to be

$$\psi + \psi_1 L_6 + \psi_2 L_6^2 + \psi_3 L_6^3 + \psi_4 L_6^4 + (\phi + \phi_1 L_6 + \phi_2 L_6^2)\, u_4 + f u_4^2,$$

with the condition which will be presently seen

$$f + u_5^2\phi_2 + u_5^4\psi_4 = 0.$$

Taking the point of contact with the standard curve as temporary origin, and writing the equation

$$b\pi^3 + 3b_1\pi^2\xi + 3a_2\pi\xi^2 + a\xi^3 + 3b_3\pi^2 + 6m\pi\xi + 3a_3\xi^2 + 3c_2\pi = 0,$$

we shall have

$$\left.\begin{aligned}
b \;&= \text{matrix, as above}\\
3b_1 &= -\, u_2^2 u_5 \{\psi_1 + 2\psi_2 L_6 + 3\psi_3 L_6^2 + 4\psi_4 L_6^3 + (\phi_1 + 2\phi_2 L_6)\, u_4\}\\
&\qquad + u_2^2 \{\phi + \phi_1 L_6 + \phi_2 L_6^2 + 2f u_4\}\, a_3\\
3a_2 &= \quad u_2^4 u_5^2 \{\psi_2 + 3\psi_3 L_6 + 6\psi_4 L_6^2 + \phi_2 u_4\}\\
&\qquad + u_2^4 (\phi + \phi_1 L_6 + \phi_2 L_6^2 + 2f u_4) - u_2^4 u_5 (\phi_1 + 2\phi_2 L_6)\, a_3 + u_2^4 f a_3^2\\
a \;&= -\, u_2^6 u_5^3 (\psi_3 + 4\psi_4 L_6) - u_2^6 u_5 (\phi_1 + 2\phi_2 L_6) + u_2^6 (2f + u_5^2\phi_2)\, a_3\\
&\text{with the condition}\\
&\qquad\qquad f + u_5^2\phi_2 + u_5^4\psi_4 = 0.\\
&\text{Also}\\
&\qquad 3b_3 = -\, u_2^3 (\phi + \phi_1 L_6 + \phi_2 L_6^2 + 2f u_4)\\
&\qquad 6m = \quad u_2^5 u_5 (\phi_1 + 2\phi_2 L_6) - 2u_2^5 f a_3\\
&\qquad 3a_3 = -\, u_2^7 u_5^2 \phi_2 - 2u_2^7 f\\
&\qquad 3c_2 = \quad u_2^6 f
\end{aligned}\right\} \quad (45).$$

We may now form the coefficient of the highest power of γ in the equation to the reciprocal curve. This coefficient is to be found, as has been shown, from the discriminant of the terms of the highest degree in this cubic, regarded as a binary cubic, and is therefore

$$b^2a^2 + 4\,ba_2^3 + 4\,b_1^3a - 6ab\,b_1a_2 - 3\,b_1^2a_2^2.$$

From this the equation to the reciprocal curve would be developed according to the previous rules but with an interchange of the operators Ω_1 and Ω_2. All the quantities in the expanded coefficient are, however, solutions of $\Omega_2 f = 0$, except a_3, and hence the degree of the reciprocal curve is indicated by the power of a_3 in the expanded expression.

25. *The matrix of the non-singular cubic, and its differential equation.*

The condition of intersecting the standard curve at a number of points coincident with the temporary origin is an invariantal relation, and in finding the condition we may treat the expressions as if they contained only differential invariants in their coefficients.

Thus we write the equation to the cubic

$$\begin{gathered}\psi\pi^3 - u_2^2u_5\psi_1\pi^2\xi + u_2^4(\phi + u_5^2\psi_2)\,\pi\xi^2 - u_2^6u_5(\phi_1 + u_5^2\psi_3)\,\xi^3 \\ - u_2^3\phi\pi^2 + u_2^2u_5\phi_1\pi\xi - u_2^7(2f + u_5^2\phi_2)\,\xi^2 + u_2^6f\pi = 0,\end{gathered}$$

and, for the consecutive points on the standard curve, putting

$$\xi = h,$$

we have (33)

$$\pi = u_2h^2 + \frac{u_5}{u_2^2}h^5 + \frac{U_7}{u_2^4u_5}h^7 + \frac{U_8 + 10u_5^4}{u_2^5u_5^2}h^8 + \frac{u_5^2U_9 - 3U_7^2}{u_2^6u_5^3}h^9 + \&\text{c.}$$

Equating to zero the coefficients of the several powers of h, on substitution for π and ξ in the equation to the cubic, we get

$$\left.\begin{array}{llll}\text{from the coefficient of } & h^2, & \psi_4 = 0 & \text{and } f + u_5^2\phi_2 = 0 \\ \quad ,, \qquad ,, & h^3, & \psi_3 = 0 & \\ \quad ,, \qquad ,, & h^4, & \psi_2 = 0 & \\ \quad ,, \qquad ,, & h^5, & \psi_1 = f & \\ \quad ,, \qquad ,, & h^6, & \psi = -u_5^2\phi_1 & \\ \quad ,, \qquad ,, & h^7, & u_5^2\phi = U_7f & \\ \quad ,, \qquad ,, & h^8, & u_5^2U_7\phi_1 = -V_8f & \end{array}\right\} \quad . \; . \quad (46).$$

These are sufficient to determine uniquely the coefficients in the matrix, and from the coefficient of h^9 we derive the differential equation to the non-singular cubic

$$\frac{u_5^2 U_9 - 3U_7^2}{u_5^3} f + \frac{V_8 + 2u_5^4}{u_5} \phi_1 - \frac{U_7}{u_5} \phi + 3u_5 \psi = 0,$$

which becomes

$$u_5^2 U_7 U_9 - V_8^2 - 4U_7^3 + u_5^4 V_8 = 0,$$

and is identical with that previously found in (36).

And the matrix of the non-singular cubic is

$$U_7 u_4^2 + \frac{U_7^2 - V_8 L_6 - U_7 L_6^2}{u_5^2} u_4 + V_8 + U_7 L_6.$$

26. *The matrix of the equation to the tangents to the cubic from the temporary origin.*

To find this equation, put $\pi = m\xi_1$, and form the discriminant of the resulting expression in ξ as a binary quantic. Equating this to zero, replace m by π/ξ. Since we need only the coefficient of the highest power of π, the simplest method of finding it is to put $\xi = 0$ in the matrix of the cubic, and form the discriminant of the resulting expression in π.

The matrix required is

$$9b_3^3 - 12bc_2,$$

or

$$(\phi + \phi_1 L_6 + \phi_2 L_6^2)^2 - 4f(\psi + \psi_1 L_6 + \psi_2 L_6^2 + \psi_3 L_6^3 + \psi_4 L_6^2) \quad . \quad . \quad (47),$$

and the degree of this in L_6 denotes the number of tangents which can be drawn.

The conditions that the cubic may be nodal or cuspidal are that this matrix, as a function of L_6, may have a linear factor twice or three times repeated.

27. *Case of nodal cubic.*

In this case we still determine uniquely all the ratios of the coefficients, except ψ/f, by the condition that the coefficients of h up to the seventh vanish identically, while the differential equation is found by equating the coefficient of h^8 to zero.

The further condition, to determine ψ/f, is (47) that the discriminant of

$$(fL_6^2 + \psi L_6 - U_7 f)^2 - 4u_5^4 f (fL_6 + \psi)$$

may vanish.

Writing k or $\psi/2f$, and x for $L_6 + k$, this equation becomes

$$x^4 - 2(k^2 + U_7)x^2 - 4u_5^4 x + (k^2 + U_7)^2 - 4u_5^4 k = 0 \quad . \quad . \quad . \quad (48).$$

With the usual notation (I and J) of theory of equations

$$\frac{3\mathrm{I}}{4} = (k^2 + \mathrm{U}_7)^2 - 3u_5{}^4k,$$

$$-\frac{27\mathrm{J}}{8} = (k^2 + \mathrm{U}_7)^3 - \tfrac{9}{2}u_5{}^4k\,(k^2 + \mathrm{U}_7) + \tfrac{27}{8}u_5{}^4,$$

and the equation giving k is

$$\{(k^2 + \mathrm{U}_7)^2 - 3u_5{}^4k\}^3 - \{(k^2 + \mathrm{U}_7)^3 - \tfrac{9}{2}u_5{}^4k\,(k^2 + \mathrm{U}_7) + \tfrac{27}{8}u_5{}^8\}^2 = 0,$$

a biquadratic for k.

To find the differential equation we have

$$\mathrm{U}_7\psi = \mathrm{V}_8 f,$$

and therefore

$$\begin{aligned}&\{(\mathrm{V}_8{}^2 + 4\mathrm{U}_7{}^3)^2 - 24u_5{}^4\mathrm{V}_8\mathrm{U}_7{}^3\}^3 \\ &\qquad - \{(\mathrm{V}_8{}^2 + 4\mathrm{U}_7{}^3)^3 - 36u_5{}^4\mathrm{V}_8\mathrm{U}_7{}^3\,(\mathrm{V}_8{}^2 + 4\mathrm{U}_7{}^3) + 216u_5{}^8\mathrm{U}_7{}^6\}^2 = 0 \quad . \quad (49),\end{aligned}$$

which contains the factor $u_5{}^{12}$.

The biquadratic from which k is to be found is

$$\mathrm{U}_7k^4 - \tfrac{1}{2}u_5{}^4k^3 + 2\mathrm{U}_7{}^2k^2 - \tfrac{9}{2}u_5{}^4\mathrm{U}_7k + \mathrm{U}_7{}^3 + \tfrac{27}{16}u_5{}^8 = 0 \quad . \quad . \quad . \quad (50).$$

The discriminant of this is found to be

$$16u_5{}^8\,\{256\mathrm{U}_7{}^3 - 27u_5{}^8\}^3.$$

And if $256\mathrm{U}_7{}^3 - 27u_5{}^8 > 0$, all the roots are real.

In this case, four real nodal cubics can be drawn to have contact of the seventh order with the standard curve. If $256\mathrm{U}_7{}^3 - 27u_5{}^8 < 0$ only two. If $256\mathrm{U}_7{}^3 - 27u_5{}^8 = 0$, there will be two nodal cubics, the third being cuspidal, as will appear in the next section.

Equiharmonic and harmonic cubics can also be drawn to have contact of the seventh order, and it is readily seen from the Theory of Forms that k will in these cases be determined by

$$(k^2 + \mathrm{U}_7)^3 - \tfrac{9}{2}u_5{}^4k\,(k^2 + \mathrm{U}_7) + \tfrac{27}{8}u_5{}^8 = 0 \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and

$$(k^2 + \mathrm{U}_7)^2 - 3u_5{}^4k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

respectively,

the roots in the latter case being all real, provided

$$256U_7^3 - 243u_5^8 > 0.$$

28. *Case of cuspidal cubic.*

The ratio ϕ/f is now to be determined from the equations of condition, as well as ψ/f.

Writing $u_5^2\phi/f = q$, we may write down the conditions from that obtained for the nodal cubic, viz.,

$$(k^2 + q)^2 - 3u_5^4k = 0.$$

$$(k^2 + q)^3 - \tfrac{9}{2}u_5^4k(k^2 + q) + \tfrac{27}{8}u_5^8 = 0.$$

Whence,

$$\left.\begin{aligned} 256q^3 &= 27u_5^8 \\ 16k^3 &= 27u_5^4 \end{aligned}\right\} \qquad (53),$$

and the differential equation becomes

$$256U_7^3 - 27u_5^8 = 0 \qquad (54).$$

29. *To exemplify the use of the general forms of the equation to the cubic and of the matrix.*

The coordinates of the tangential of the origin are

$$\pi = 0, \quad \xi = -\frac{3a_3}{a},$$

and the matrix of the polar conic of the tangential becomes

$$(\phi_1 + 2\phi_2L_6)[fu_4 + (\phi + \phi_1L_6 + \phi_2L_6^2)] - f\psi_1.$$

Taking the general cubic this becomes

$$\frac{(V_8 + 2U_7L_6)}{u_5}\left[U_7u_4 + \frac{(U_7^2 - V_8L_6 - U_7L_6^2)}{u_5^2}\right] + u_5,$$

and the matrix (and conic) breaks up into factors, provided $u_5 = 0$, since $U_7^2 - V_8L_6 - U_7L_6^2$ contains the factor u_5^2.

That is, as is known, the tangent at a sextactic point passes through a point of inflexion.

The matrix of the nodal cubic is given by

$$b/u_5^2 = 2k + L_6 + \frac{U_7 - 2kL_6 - L_6^2}{u_5^2}u_4 + u_4^2.$$

The biquadratic for k may be written

$$\mathrm{U}_7 (k^2 + \mathrm{U}_7)^2 = \tfrac{1}{2} u_5^4 (k^3 + 9\mathrm{U}_7 k - \tfrac{27}{8} u_5^4),$$

and $k^2 + \mathrm{U}_7$ contains the factor u_5^2.

Hence, from the equation (48) for x, x or $k + \mathrm{L}_6$ contains the factor u_5, and $\mathrm{U}_7 - 2k\mathrm{L}_6 - \mathrm{L}_6^2$ or $\mathrm{U}_7 + k^2 - (k + \mathrm{L}_6)^2$ contains the factor u_5^2, so that u_5^2 is a factor of the matrix b.

The matrix of the polar conic of the tangential of the temporary origin is therefore

$$2\frac{k + \mathrm{L}_6}{u_5}\left\{u_4 + \frac{\mathrm{U}_7 - 2k\mathrm{L}_6 - \mathrm{L}_6^2}{u_5^2}\right\} + u_5,$$

and the tangent at a sextactic point will pass through the point of inflexion.

Evidently there can not exist any sextactic point on the cuspidal cubic.

§ 5. *Invariant Coordinates and Invariantal Equations.*

To make it possible to use the methods of this paper to investigate the properties of the higher plane curves, it is desirable to effect some gain in brevity and lucidity without any sacrifice in generality. This is the object of the system of invariant coordinates, which are introduced in this last section.

The matrix from which the equation to a covariant curve is developed contains u_4, L_6, and invariant functions. The order of the curve, in general, depends upon the mode in which u_4 and L_6 enter the matrix, and its characteristics upon the ratios between the invariant functions. These ratios have the properties which entitle them to be regarded as the invariantal coordinates of the curve whose equation is developed from the matrix. In especial, I shall show that this is the case in the equation to a covariant straight line, or line of homographic persistence, and, from the consideration of such lines, arrive at a definition of invariant coordinates of a point of homographic persistence. After showing that we obtain a dual invariant system of point and line coordinates, I ultimately develop a mode of representing a curve by an invariant equation, which will be much shorter than the covariant form of the equation which we have as yet considered. The coordinates to the curve will be the characteristic invariants of the curve, and none of the homographic peculiarities will be sacrificed by the abbreviation.

30. *Introduction of invariant coordinates of a homographically persistent point or curve, and of an intrinsic invariant equation to such a curve.*

The general equation to a covariant line takes the form

$$(u_5{}^2\phi_3 u_4 + \phi_1 + \phi_2 \mathrm{L}_6 - \phi_3 \mathrm{L}_6{}^2)\,\pi - u_2{}^2\,\{(\phi_2 - 2\phi_3 \mathrm{L}_6)\,u_5 - u_5{}^2\phi_3 \alpha_3\}\,\xi - u_2{}^3 u_5{}^2 \phi_3 = 0 \ldots$$

where all the functions ϕ contain differential invariants only. Such a line may be called homographically persistent with regard to the standard curve.

Indicating a second homographically persistent line by a similar notation, but using dashed functions, we may find the coordinates of the point of intersection in the form

$$\left.\begin{aligned} \pi &= u_2{}^3\mathrm{A}/u_4\mathrm{A} + \mathrm{C} + \mathrm{B}\alpha_3 \\ \xi &= u_2\mathrm{B}/u_4\mathrm{A} + \mathrm{C} + \mathrm{B}\alpha_3 \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (55),$$

where

$$\left.\begin{aligned} \mathrm{A} &= -\,u_5{}^2\,(\phi_2\phi_3' - \phi_2'\phi_3) \\ \mathrm{B} &= u_5\,\{(\phi_3\phi_1' - \phi_3'\phi_1) - (\phi_2\phi_3' - \phi_2'\phi_3)\,\mathrm{L}_6\} \\ \mathrm{C} &= (\phi_1\phi_2' - \phi_1'\phi_2) + 2\,(\phi_3\phi_1' - \phi_3'\phi_1)\,\mathrm{L}_6 - (\phi_2\phi_3' - \phi_2'\phi_3)\,\mathrm{L}_6{}^2 \end{aligned}\right\},$$

or, more shortly,

$$\left.\begin{aligned} \mathrm{A} &= -\,u_5{}^2\lambda \\ \mathrm{B} &= u_5\,(\mu - \lambda\mathrm{L}_6) \\ \mathrm{C} &= \nu + 2\mu\mathrm{L}_6 - \lambda\mathrm{L}_6{}^2 \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (56).$$

We may call such a point a point of homographic persistence with regard to the standard curve.

Treating $(\lambda : \mu : \nu)$ as determining the position of a point, and $(\phi_1 : \phi_2 : \phi_3)$ the position of a line, the condition that $(\lambda : \mu : \nu)$ may lie on $(\phi_1 : \phi_2 : \phi_3)$ is

$$\lambda\phi_1 + \mu\phi_2 + \nu\phi_3 = 0.$$

Hence these form correlative systems of point- and line-coordinates.

If two lines $(\phi_1 : \phi_2 : \phi_3)$, $(\phi'_1 : \phi'_2 : \phi'_3)$ meet in $(\lambda : \mu : \nu)$

$$\frac{\lambda}{\begin{vmatrix} \phi_2, & \phi_3 \\ \phi'_2, & \phi'_3 \end{vmatrix}} = \frac{\mu}{\begin{vmatrix} \phi_3, & \phi_1 \\ \phi'_3, & \phi'_1 \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} \phi_1, & \phi_2 \\ \phi'_1, & \phi'_2 \end{vmatrix}}$$

and if two points $(\lambda : \mu : \nu)$, $(\lambda' : \mu' : \nu')$ lie on $(\phi_1 : \phi_2 : \phi_3)$

$$\frac{\phi_1}{\begin{vmatrix} \mu, & \nu \\ \mu', & \nu' \end{vmatrix}} = \frac{\phi_2}{\begin{vmatrix} \nu, & \lambda \\ \nu', & \lambda' \end{vmatrix}} = \frac{\phi_3}{\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}}$$

If three points $(\lambda : \mu : \nu)$, $(\lambda' : \mu' : \nu')$, $(\lambda'' : \mu'' : \nu'')$ lie on a straight line

$$\begin{vmatrix} \lambda, & \mu, & \nu \\ \lambda', & \mu', & \mu' \\ \lambda'', & \mu'', & \nu'' \end{vmatrix} = 0,$$

and so on.

I define $(\lambda : \mu : \nu)$ as the invariantal coordinates of the point, and $(\phi_1 : \phi_2 : \phi_3)$ as the invariantal coordinates of the line.

The condition that $(\lambda : \mu : \nu)$ may lie upon a covariant curve of the n^{th} order will be an invariantal relation between λ, μ, ν and the coordinates of the curve, homogeneous of the n^{th} degree in λ, μ, ν.

It follows that if $f(\lambda, \mu, \nu) = 0$ expresses this relation, it is, in this system of coordinates, the equation of the curve. I shall call it the intrinsic invariant equation to the curve.

The coordinates of the tangent to the curve at $(\lambda : \mu : \nu)$ are

$$\frac{\partial f}{\partial \lambda} : \frac{\partial f}{\partial \mu} : \frac{\partial f}{\partial \nu}.$$

If $(\lambda' : \mu' : \nu')$ lies on the tangent to the curve at $(\lambda : \mu : \nu)$ then

$$\lambda' \frac{\partial f}{\partial \lambda} + \mu' \frac{\partial f}{\partial \mu} + \nu' \frac{\partial f}{\partial \nu} = 0.$$

Taken with $f(\lambda, \mu, \nu) = 0$, this gives the coordinates of the points of contact of tangents from $(\lambda' : \mu' : \nu')$, and, being of the $n - 1^{\text{th}}$ degree, it represents the first polar of $f(\lambda, \mu, \nu) = 0$ at $(\lambda' : \mu' : \nu')$.

Finding the first polar of this again, the second polar of $f(\lambda, \mu, \nu) = 0$ has the equation

$$\left(\lambda' \frac{\partial}{\partial \lambda} + \mu' \frac{\partial}{\partial \mu} + \nu' \frac{\partial}{\partial \nu}\right)^2 f = 0.$$

It is unnecessary to carry this further to determine how far the ordinary methods of geometry apply to this novel system of coordinates, since the character of the analogy is obvious.

31. *The values of the differential invariants at different points on the curve.*

The coordinates of any point on a covariant curve were, (55) and (56), given in the form

$$\pi = - u_2{}^3 u_5{}^2 \mathrm{Y} \,/ - (u_4 u_5{}^2 + \mathrm{L}_6{}^2 + u_5 \mathrm{L}_6 a_3)\, \mathrm{Y} + (\mathrm{L}_6 + u_5 a_3)\, \mathrm{X} + 1,$$

$$\xi = - u_2 u_5 (\mathrm{L}_6 \mathrm{Y} - \mathrm{X}) \,/ - (u_4 u_5{}^2 + \mathrm{L}_6{}^2 + u_5 \mathrm{L}_6 a_3)\, \mathrm{Y} + (\mathrm{L}_6 + u_5 a_3)\, \mathrm{X} + 1,$$

where X and Y have been written for μ/ν and λ/ν respectively. Here X and Y are the variables as we pass along a curve, and $d^2\mathrm{Y}/d\mathrm{X}^2$, &c., are to be found from the intrinsic invariant equation to the curve which we may write $f(\mathrm{X}, \mathrm{Y}) = 0$.

The relations between ξ, π and X, Y are of the form of a homographic transformation, and therefore any of the functions of $d^2\pi/d\xi^2$, &c., which we know to be differential invariants, will only differ from the similar functions of $d^2\mathrm{Y}/d\mathrm{X}^2$, &c., by a factor, of which we know the form, which will contain u_4, L_6, and a_3.

It will not be possible to express the value of a differential invariant at (X, Y) in terms of X, Y and differential invariants at the origin only, unless the said differential invariant is absolute, but the invariantal coordinates of a point of homographic singularity will be found by applying the condition immediately to $f(\mathrm{X}, \mathrm{Y}) = 0$.

Obviously, if the intrinsic invariantal equation is regarded as the equation to a curve in the ordinary sense, that curve will have the same homographic singularities as the covariant from which it is derived, and the coefficients in the equation are differential invariants. Hence we are led to the simpler method of finding the conditions for the existence of homographic singularities in terms of differential invariants, namely, the immediate application of the ordinary theory of forms to the intrinsic invariantal equation treated as a ternary quantic. For this reason, the equation in this form may be regarded as the canonical form of the equation to the curve, since the coefficients are the differential invariants which characterise the curve.

I proceed to illustrate this upon the conic and cubic.

To obtain the intrinsic invariant equation, we may omit all terms which are not purely invariant from the coefficients in the covariant form of the equation and substitute $-u_2^3u_5^2\lambda/\nu$ or $-u_2^3u_5^2\mathrm{Y}$ for π, and $u_2u_5\mu/\nu$ or $u_2u_5\mathrm{X}$ for ξ. I shall retain λ, μ, ν.

Thus, the intrinsic invariant equation to the osculating conic is

$$\lambda\nu + \mu^2 = 0. \qquad \qquad (57).$$

32. *Illustrations in the case of the cubic.*

The intrinsic invariantal equation to the general non-singular cubic is (omitting a factor containing u_2)

$$u_5^{\,4}\lambda^2(\mathrm{V}_8\lambda + \mathrm{U}_7\mu) + (\mathrm{U}_7^{\,2}\lambda - \mathrm{V}_8\mu + \mathrm{U}_2\nu)(\lambda\nu + \mu^2) = 0 \quad . \quad . \quad . \quad (58),$$

and that of its Hessian is consequently

$$\begin{vmatrix} 6u_5^{\,4}\mathrm{V}_8\lambda+2u_5^{\,4}\mathrm{U}_7\mu+2\mathrm{U}_7^{\,2}\nu, & 2u_5^{\,4}\mathrm{U}_7\lambda+2\mathrm{U}_7^{\,2}\mu-\mathrm{V}_8\nu, & 2\mathrm{U}_7^{\,2}\lambda-\mathrm{V}_8\mu+2\mathrm{U}_7\nu \\ 2u_5^{\,4}\mathrm{U}_7^{\,2}\lambda+2\mathrm{U}_7^{\,2}\mu-\mathrm{V}_8\nu, & 2\mathrm{U}_7^{\,2}\lambda-6\mathrm{V}_8\mu+2\mathrm{U}_7\nu, & -\mathrm{V}_8\lambda+2\mathrm{U}_7\mu \\ 2\mathrm{U}_7^{\,2}\lambda-\mathrm{V}_8\mu+2\mathrm{U}_7\nu, & -\mathrm{V}_8\lambda+2\mathrm{U}_7\mu, & 2\mathrm{U}_7\lambda \end{vmatrix} = 0.$$

If $(\lambda_1 : \mu_1 : \nu_1)$ lies on the tangent at the origin, $U_7\lambda_1 = 0$, and for the tangential of the origin $\lambda_1 = 0$, $\mu_1 : \nu_1 = U_7 : V_8$.

Hence, on a non-singular cubic we can have neither $U_7 = 0$ nor $V_8 = 0$.

Also, on substitution in the equation to the Hessian, we obtain $u_5 = 0$, or, if the tangential is a point of inflexion, the origin must be a sextactic point, as is well known.

The equation to the conic of closest contact at the origin is

$$\lambda\nu + \mu^2 = 0,$$

and to the polar conic of the origin is

$$U_7^2\lambda^2 - V_8\lambda\mu + 2U_7\lambda\nu + U_7\mu^2 = 0.$$

So that

$$U_7^2\lambda - V_8\mu + U_7\nu = 0$$

is the equation to the common chord of these two conics, and it is the tangent at the tangential of the origin.

The second tangential of the origin lies on

$$V_8\lambda + U_7\mu = 0,$$

that is, lies on the common chord of the cubic and the conic of the closest contact.

The coordinates of the point at which this chord meets the cubic again, or the sixth point in which the osculating conic meets the cubic, are given by

$$\frac{\lambda}{U_7^2} = \frac{\mu}{-U_7V_8} = \frac{\nu}{-V_8^2},$$

and, therefore, $V_8\mu - U_7\nu = 0$ is the equation to the line joining this point to the tangential of the origin.

The cordinates of the third tangential of the origin are given by

$$\frac{\lambda}{U_7^2} = \frac{\mu}{-u_5^4U_7} = \frac{\nu}{-(U_7^3 + u_5^8)},$$

and are independent of V_8. This point is the corresidual of the eight consecutive points on each of the several cubics in which V_8 is arbitrary. This is also a known property, but the method allows us, with little difficulty, to prove properties of which other analytical proofs are laborious.

33. *General method of finding the form of the intrinsic invariant equation to a curve of any order.*

It will not be necessary to pass through all the steps which I have taken in developing this theory of intrinsic invariant equations.

If we have an equation to a covariant curve, say $f(\pi, \xi) = 0$, and if $\mathrm{F}(\lambda:\mu:\nu) = 0$, or $\mathrm{F}(\mathrm{X}, \mathrm{Y}) = 0$, is the corresponding intrinsic invariant equation where X and Y stand for μ/ν and λ/ν respectively, then the relations between π, ξ and X, Y are essentially of the character of a homographic transformation. Hence, if for points near the origin in the covariant curve π is written $a_2h^2 + a_5h^5 + a_7h^7 + \ldots$, and h is put for ξ, while in the intrinsic invariant equation Y is written $\mathrm{A}_2h^2 + \mathrm{A}_5h^5 + \mathrm{A}_7h^7 + \ldots$, where h is put for X, and if a_2, a_5, a_7, &c., are written for the invariantive portions of a_2, a_5, a_7, &c., then A_2, A_5, A_7 differ from the values of a_2, a_5, a_7, &c., by factors of the character which we have considered in the earlier part of this paper.

Thus $\mathrm{a}_n = \mathrm{D}p^{2n-1}q^{-n-1}\mathrm{A}_n$, where p and q stand for the expressions corresponding to those written μ and λ in (2).

In general $\mathrm{D} = -u_2^4u_5^3$, and at the origin

$$q = u_2u_5, \quad p = 1.$$

Therefore,

$$\mathrm{a}_n = -u_2^4u_5^3(u_2u_5)^{-n-1}\mathrm{A}_n, \text{ or } \mathrm{A}_n = -u_2^{n-3}u_5^{n-2}\mathrm{a}_n,$$

therefore

$$\begin{aligned} \mathrm{A}_2 &= -1 \\ \mathrm{A}_5 &= -u_5^4 \\ \mathrm{A}_7 &= -u_5^4\mathrm{U}_7 \\ \mathrm{A}_8 &= -u_5^4\mathrm{W}_8, \text{ where } \mathrm{W}_8 = \mathrm{V}_8 + 2u_5^4, \end{aligned}$$

and, comparing with (33), generally

$$\mathrm{A}_n = -u_5^4\mathrm{W}_n.$$

Hence the value which, near the origin, is to replace Y is

$$-h^2 - u_5^4(h^5 + \mathrm{U}_7h^7 + \mathrm{W}_8h^8 + \ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

Taking the general equation of the intrinsic invariant of conics as

$$\theta_5\lambda^2 + \theta_4\lambda\mu + (\theta_3 + \theta_2)\mu^2 + \theta_2\lambda\nu + \theta_1\mu\nu + \theta\nu^2 = 0,$$

and substituting for λ/ν and μ/ν, the values shown above, we find all the invariant coefficients vanish except θ_2, and thus

$$\lambda\nu + \mu^2 = 0$$

is, as before, the intrinsic invariant equation of osculating conics.

So we may, without further investigation, find the similar equation for osculating cubics as we have found it above.

The equation to the osculating curve of each order will aid us in finding that of the next; thus, the general form of a non-singular osculating quartic will be

$$\theta_1\lambda\,(u_5{}^4\lambda^3 - \lambda\mu\nu - \mu^3) + \theta_2\lambda^2\,(\lambda\nu + \mu^2) + \theta_3\,(\lambda\nu + \mu^2)^2$$
$$+ (\theta_4\lambda + \theta_5\mu + \theta_6\nu)\{u_5{}^4\lambda^2\,(\mathrm{V}_8\lambda + \mathrm{U}_7\mu) + (\mathrm{U}_7{}^2\lambda - \mathrm{V}_8\mu + \mathrm{U}_7\nu)\,(\lambda\nu + \mu^2)\} = 0.$$

The forms of the functions θ can be found, but they depend upon differential invariants of a higher order than those of which the values have been investigated.